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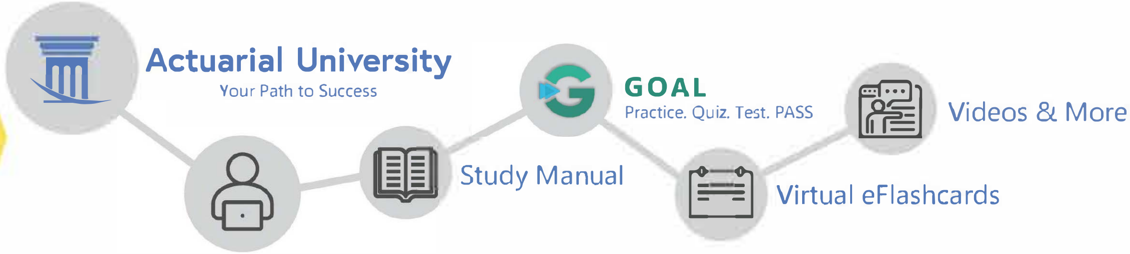

a/s/m
Exam FM Study Manual

16th Edition

Harold Cherry, FSA, MAAA and Wafaa Shaban, ASA, Ph.D.

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Exam FM Study Manual

16th Edition

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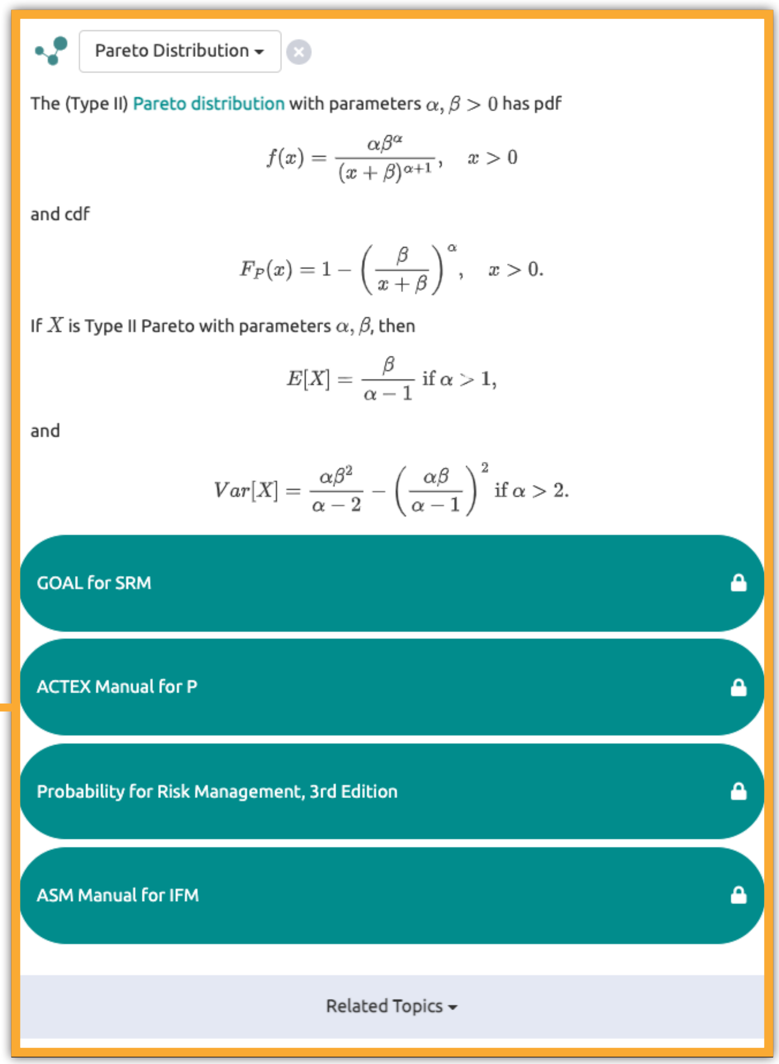

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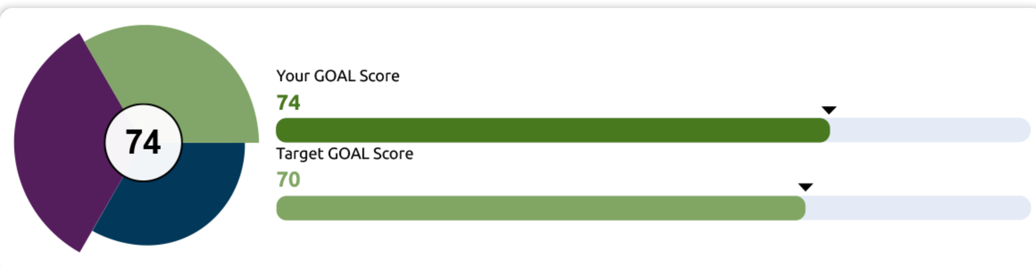

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62	74	78
General Probability	Random Variables with Univariate Probability Distributions	Random Variables with Univariate Probability Distributions
Weighting: 13.4%	Weighting: 43.3%	Weighting: 43.3%

See key areas where you can improve.

General Probability — Set functions including set notation and basic elements of probability

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Core	58	92 / 304
Advanced	60	169 / 304
Mastery	78	43 / 304

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GOAL Session Activity Summary — Filter: All Types, All Statuses

Created	Last Accessed	Completed	Mode	Categories	Questions	Status	Status
05/24/2022 21:57:43	05/24/2022 21:57:43		Quiz	Continuous P...	25	New	Resume
05/24/2022 10:34:05	05/24/2022 14:57:49	05/24/2022 14:57:49	Practice Session	Addition and ...	80	Complete	Review
05/21/2022 19:32:50	05/23/2022 20:02:00		Simulated Exam	Exam 2	30	Reviewing	Complete
05/17/2022 15:19:19	05/17/2022 15:46:03	05/23/2022 14:15:29	Simulated Exam	Exam 6	30	Complete	Review
05/14/2022 11:26:59	05/14/2020 12:02:36	05/23/2022 11:57:47	Quiz	Conditional D...	20	Complete	Review

Quickly return to previous sessions.

Contents

Introduction

To the student: Please read this Introduction. It contains important information.

The 16^{th} edition of this manual has been revised in accordance with the new SOA syllabus beginning with the October 2022 exam. Also, this edition incorporates corrections of all known errata in the 15^{th} edition.

The 16^{th} edition consists of 9 chapters covering all of the material on the syllabus (Part I of the manual), followed by six original practice exams (Part II of the manual). Following each chapter or section in Part I, there are illustrative examples called "Stepping Stones" (see below). These examples are followed by a summary of the key concepts and formulas in the section, then by problems and solutions from actual past SOA/CAS examinations.

Goals of this Manual

- To explain the concepts of financial mathematics in a way that appeals to your intuition and common sense.
- To point out shortcuts and tricks that can get you to the answer more quickly.
- To warn you about common traps that students fall into and help you to avoid them.
- To provide you with hundreds of problems from past exams, with solutions.
- To provide you with original practice exams that will help prepare you for the real thing.

To highlight the concepts, tricks, shortcuts, and traps, you will see special symbols such as the following throughout the manual:

Concept Alert! **Shortcut Alert!**

Trick Alert! **Trap Alert!**

Problems

There is an old cliché in the real estate industry: What are the three most important factors in evaluating property? Answer: Location! Location! Location! If we had to say what the three most important factors are in passing this exam, they would be Problems! Problems! Problems! You must do a great variety of problems, preferably under time pressure, and especially as you get closer to the date of the examination.

Many of the problems in this manual are taken from past SOA and CAS examinations. We want to thank the two actuarial societies for their kind permission to publish these questions.

There are a number of points about past exam questions that you should be aware of:

- These questions, which date from the early 1980's, were created by different exam committees, under different syllabi, for exams of different length, etc. Thus, they can vary greatly in style, difficulty and emphasis of topics. In spite of this, you should find that solving prior exam problems, even very old ones, is helpful in preparing for the exam. It will expose you to a great variety of types of questions and approaches to solutions.
- This manual does not contain the questions and solutions from past FM exams published by the SOA. Currently, only the May 2005 and November 2005 exams have been published (but note that these two exams were very easy and followed an old syllabus that was in effect in 2005). You will also find a link to well over 200 sample questions and solutions, many of which are from past exams. ***You should definitely get these additional practice problems.*** You

can download them for free from the SOA website under Syllabus. We have not included them in this manual because we didn't want to waste space and expense by duplicating material that is freely available from another source.

- Since October 2022, the actual exam consists of 30 questions in CBT (computer based testing) format, with a $2\frac{1}{2}$-hour time limit.
- The order of the past exam questions that follow each topic in this manual is from the most recent to the oldest.
- There is a code following each question in this manual from a past exam that tells you which exam the question came from.
- Some of the questions have answer choices in ranges, such as "(A) Less than 11.3% (B) At least 11.3% but less than 11.5%," etc. These questions almost always come from CAS exams, which used this style for many years. Virtually all questions on the SOA/CAS Exam FM/2 since 2000 have had specific answer choices, rather than ranges.

To make it easier to locate them, the sections of the manual with past exam questions and solutions have running heads like this:

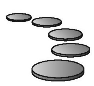
Past Exam Questions on Sections 1b to 1f

and

Solutions to Past Exam Questions on Sections 1b to 1f

"Stepping Stones"

Following each section of the manual, you will find illustrative examples, with solutions, that are called "Stepping Stones." They are indicated by this heading:



Stepping Stones

These examples are original to the manual and have a level of difficulty that runs from easy to moderate. They have two purposes:

- To make sure you understand the material that you have just read.
- To serve as "stepping stones" to more difficult exam-level problems.

Original Practice Exams

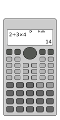
In addition to the questions from past exams, the manual contains six original full-length practice exams, with solutions. These exams consist of 30 questions each, with a $2\frac{1}{2}$-hour time limit, just like the actual exam. The distribution of questions by topic follows the SOA syllabus.

Take these exams in the order given. We believe that Practice Exam 6 is the most difficult. As we receive feedback from students about the difficulty of the six exams, we will post the information on the ASM website at www.studymanuals.com.

Calculators

Begin using your calculator immediately. Become thoroughly familiar with its operation. It should become like a trusted friend to you after awhile.

Our advice is to get a financial calculator for the exam, such as the BA II Plus© or BA II Plus Professional©. These calculators have special keys that are very useful for solving problem in financial mathematics. Which one should you use? You really don't need the Professional model, unless you are willing to spend extra money for some additional features that aren't essential for the exam. For this reason, we have based the Calculator Notes in this manual on the BA II Plus. You will find these notes at appropriate points in the manual, identified by this icon:



Please note that many students prefer to use one of the TI-30 MultiView series as their primary calculator because of the two-line display and superior algebraic capabilities. These students feel that they can even calculate certain interest functions, such as $a_{\overline{n}|}$, more quickly on the TI-30 series, even though the BA series has special keys for this function. They use the BA series as a backup calculator, mainly to calculate an unknown interest rate when the amounts of payments and their present value are given.

For a quick reference, here is the list of the Calculator Notes along with their corresponding page numbers:

Summary of Concepts

Following most sections in Part I, there is a summary of the key concepts and formulas covered in the section. To make it easier to locate them and for a quick review before taking your exam, the Summary of Concepts along with their corresponding page numbers are listed in the table below.

Studying for the Exam

Everyone is different. Some people like to study in the morning, some late at night. Some people can only study for a couple of hours before their brain begins to fry; others can study for long stretches. Some people can cram very effectively, but others need to cover the material over a period of several months. Almost everyone finds ways to procrastinate from time-to-time. You know your own study habits best, so build on your strengths and be aware of your weaknesses.

We recommend that you set up a study schedule right now. Determine how many hours you will realistically be able to put in between now and the exam. Many students use this rule-of-thumb: devote 100 hours of study time for each hour of exam time. For a $2\frac{1}{2}$-hour exam like FM/2, this would mean 250 hours. Of course, people vary considerably in how quickly they learn new material, so use this rule-of-thumb only as a rough guide.

When you set up your study schedule, be very aware of the SOA's allocation of questions by topic. The following table shows how many questions, on the average, you should expect for each topic, assuming that the SOA sticks to its own allocation when it constructs your particular computer-based exam. The table also shows which sections of the manual cover each topic.

Learning Objective	**Allocation by SOA**	**Midpoint of Allocation**	**Number of Questions (Based on a 30-Question Exam)**	**Chapters or Sections of the Manual that Cover the Topic**
1. Time Value of Money (Interest rate, discount rate, nominal rate, effective rate, force of interest, equation of value, inflation and real rate of interest, etc.)	5–15%	10%	3	Chapters 1, 2, 5
2. Annuities (annuity-immediate and due, perpetuity, m-thly, continuously, level payments, arithmetic and geometric increasing/decreasing, etc.)	20–30%	25%	7.5	Chapters 3, 4
3. Loans (Principal, interest, outstanding balance, final payment, amortization, refinancing, etc.)	15–25%	20%	6	Chapter 6
4. Bonds (Price, book value, amortization of premium, accumulation of discount, yield rate, coupon, face or par value, redemption value, etc.)	15–25%	20%	6	Chapter 7
5. General Cash Flows, Portfolios and Immunization (Yield rate, Macaulay and modified duration and convexity, spot rates, forward rates, yield curve, cash flow matching, Redington and full immunization, change in PV due to change in interest rate, etc.)	20–30%	25%	7.5	Chapters 5, 8, 9
Grand Total		100.0%	30	

The above table is based on the final syllabus for the October 2022 exam as posted by the SOA. If you are taking the exam in October 2022 or later, check the syllabus for the particular month and year of your exam, since the exam committee does make changes from time to time.

As you follow your schedule, will you fall behind from time-to-time? Of course. But if you have a schedule, at least you will know how far behind you are. This should spur you on to catch up.

Should you do all of the problems in a section before you move on to the next section? Our advice is to do as many problems as you can, but to move on if you are falling behind schedule; hopefully, you will catch up later.

You should leave room at the end of your schedule for at least three or four weeks of doing nothing but solving problems (and maybe a little bit of review of the topics you are having difficulty with). Be strict with yourself and work "by the clock." It's not important that you get the correct answer to a problem the first time that you do it. What *is* important is that you try the problem again a couple of days later and get it right the second (or even the third) time. (Mark off the problems that you don't get right and do batches of these problems a few days later.) This shows that you have really mastered the points being tested.

"Hitting a Brick Wall": Chapters 3 and 4

Some students find that when they attempt to cover the topics in Chapters 3 and 4, it's like "hitting a brick wall." There is a lot of material in these two chapters and it can be daunting if you are encountering it for the first time. If this happens to you, we have a couple of suggestions:

- Don't knock yourself out if you don't understand something. Move on to the next section and come back later on. You will find that as you spend more time on the material, you will mature in your understanding. Concepts that seemed difficult at first will seem easier when you revisit them.
- Don't try to do all of the problems at the end of each section the first time around. Do the first few, since they are the most recent, or do only the odd or even problems. Come back later and do the rest of the problems (or most of them). They will seem easier the second time around.

It's true that there are a lot of problems in these two chapters, about 200 in all, but this shows how important these topics have been on past exams.

Taking the Exam

The single most important rule about taking the exam is to *keep moving.* Don't get bogged down on any one question. Try to look at every question at least once. Be aware of your progress throughout the exam. Try to spend your time on the problems that are easier for you and drop the ones that are giving you too much trouble.

Since this exam has 30 questions and a $2\frac{1}{2}$-hour time limit, you have an average of about 5 minutes to spend on each question. You may be tempted to continue spending time on a problem because you have already invested, say, 5 or 6 minutes in it. We strongly suggest that you develop the discipline to drop the question at this point and move on. There may be an easier question waiting for you later on—and getting it right counts just as much as getting a hard question right. And you may have time to go back to the question later on and knock it off.

Your Comments

We welcome your comments, criticism, suggestions, and reports of any errata that you may find. Please e-mail us at *mail@studymanuals.com*

We wish you good studying and good luck on the exam. May the force of interest be with you.

Harold Cherry and Wafaa Shaban

Part I

Financial Mathematics

Chapter 2

Practical Applications

2a Equations of Value, Time Value of Money, and Time Diagrams

Equations of Value and the Time Value of Money

Suppose that deposits made today can earn 5% effective over the next year. Which would you rather have, $1,000 today or $1,050 in a year?

A basic principle of equivalence that we will use in interest theory is that it doesn't matter to us whether we have $1,000 today or $1,050 in a year (assuming that the effective rate of interest is 5%). In fact, if we are told that the effective rate is 5% for the next 10 years, we will assume that we would be just as happy with $1,000 today as we would be with $1,000(1.05)^t$ in t years, where t is any time from 0 to 10.

This basic principle of equivalence may seem so obvious that it doesn't even have to be stated. But we must agree to it if we want to play the game in accordance with the rules of interest theory.[1]

Another way to say this is that money has a ***time value***: $1,000 today is worth more than $1,000 a year from now. So if person A paid person B $1,000 now in return for a payment from B in a year, A would want that payment to be more than $1,000. You will often see the term ***time value of money*** in the financial literature.

The principle of equivalence allows us to solve interest problems by setting up ***equations of value*** as of a common ***comparison date***. As an example, let's say that you want to accumulate $5,000 in two years by making a deposit of X today and another deposit of X in a year. If the effective rate of interest is 6%, determine X.

To solve for X we will equate the value of the deposits and the accumulated value of $5,000. In order to do this, we must choose a common date, called the comparison date. Let's say we choose time 2. As of that date, the deposits are worth $X(1.06)^2 + X(1.06)$, by the principle of equivalence. The AV we want at that point is $5,000. Thus, the equation of value as of time 2 is:

$$X(1.06^2 + 1.06) = 5{,}000$$
$$X = \frac{5{,}000}{1.06^2 + 1.06} = \$2{,}289.80$$

Two points should be noted:

1. You ***must*** choose the same date (the comparison date) to evaluate the deposits and the AV. You cannot directly compare X deposited today and X deposited a year from now with $5,000 received two years from now, unless you determine the equivalent values of these amounts as of the same date.

2. ***Any*** date can be used for the comparison date. For example, suppose we had used a ridiculous date like 122 years from now for the comparison date. What would the equation of value be?

$$\text{Answer: } X(1.06)^{122} + X(1.06)^{121} = 5{,}000(1.06)^{120}$$

   This is the same equation of value as before, but with both sides multiplied by 1.06^{120}. Thus, we would get the same solution for X.

[1] There may be some people who would not be just as happy with $1,050 a year from now as they would be with $1,000 today, even if 5% were a fair rate of interest. (Can you think of any reasons why?) But we will assume that the principle of equivalence applies when we solve problems in this course.

Certainly, choosing time 122 would unnecessarily complicate the solution. In this example, time 0 or time 2 (and perhaps time 1) are the obvious choices. Choosing a convenient comparison date is something you will learn to do as you get more experience with solving compound interest problems.

Time Diagrams: The Student's Friend

Some people are lucky: They can visualize deposits and withdrawals, ***payment periods*** and ***interest periods***, etc., without the aid of a diagram. But for most of us, a time diagram is a very useful aid in solving problems in compound interest.

A ***time diagram*** is a statement of a problem in picture form. Once the diagram is drawn, the problem often practically solves itself.

The following examples are basic in nature. If you feel comfortable about setting up time diagrams, or if you don't need them to solve problems, you can skip this section. (But you might want to look at Examples 2.3 and 2.4 in any event.)

Stepping Stones

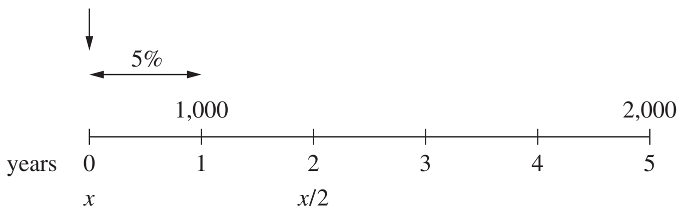
Example 2.1

Draw a time diagram and write an equation of value for the following problem:

A deposit is to be made today and a second deposit, which is one-half the first, is to be made 2 years from now, to provide for withdrawals of $1,000 one year from now and $2,000 5 years from now. Interest is at 5% effective. What is the amount of the initial deposit?

Solution

Let x = initial deposit. Then the deposit at time 2 is $x/2$.



Using time 0 as the comparison date, the equation of value is:

$$x + (x/2)v^2 = 1{,}000v + 2{,}000v^5 \quad \text{at } 5\%$$

$$x = \frac{1{,}000v + 2{,}000v^5}{1 + v^2/2} = \boxed{\mathbf{\$1{,}733.34}}$$

Note the following points:

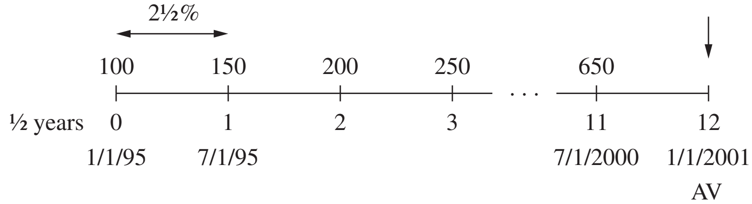
1. Deposits and withdrawals are placed on opposite sides of the time line (for example, all deposits below the line and all withdrawals above the line, as in the above diagram). This will assure that we get the correct terms on the left and right-hand sides of the equation of value.
2. To the left of the time line, the interest period is noted ("years"). In this problem, where the interest period and payment period are the same, it isn't too important but get into the habit of noting it on your diagrams anyway.
3. The effective interest rate for one interest period (5%) is marked off, as a reminder of the rate to be used in calculations.
4. A vertical arrow is placed at the comparison date chosen; in this solution, time 0 was chosen as the comparison date. Of course, any chosen date would lead to the correct value of x.

Example 2.2

Deposits are made on January 1 and July 1 of every year from 1995 to 2000, inclusive. The initial deposit is \$100 and each subsequent deposit increases by \$50. What is the accumulated value of the deposits on January 1, 2001 at a nominal rate of interest of 5% compounded semiannually?

Solution

In this problem, the effective rate of interest is $2\frac{1}{2}\%$ for a ***half-year*** period and payments are made every ***half-year***, so it's a no-brainer that we should label the diagram in terms of half-year periods.



We have marked off "$\frac{1}{2}$ years" to the left of the time line as a reminder that "time 1" is 6 months from now, "time 2" is a year from now, etc., and that the effective rate of $2\frac{1}{2}\%$ applies to each of these 6-month periods. Also, we have translated the given ***dates*** into ***interest periods***, starting with 1/1/95 as time 0 and ending with 1/1/2001 as time 12 (6 years later).

The equation of value using "brute force" (that is, term-by-term without any fancy annuity symbols) and using time 12 as the comparison date is:

$$\boxed{\mathbf{100(1.025)^{12} + 150(1.025)^{11} + 200(1.025)^{10} + \cdots + 650(1.025) = \boldsymbol{AV}}}$$

Of course, the techniques for handling varying annuities that are developed later in this manual would be much more efficient for determining the AV than a term-by-term calculation. But at this point, we are primarily interested in the time diagram as a technique for setting up problems.

Example 2.3

Draw a time diagram and write an equation of value for the following problem:

Find the present value of quarterly payments of \$100 for 10 years, first payment 3 months from now, at a nominal rate of interest of 6% compounded semiannually.

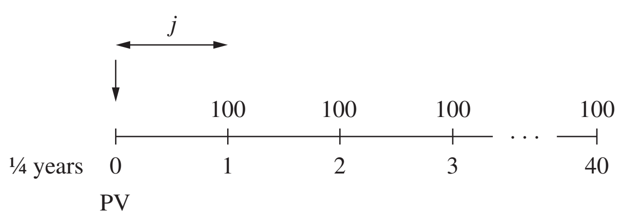
Solution

In this problem, the payment period and the interest period are not the same. The payment period is 3 months and the interest period is 6 months (since the given nominal interest rate implies an effective rate of 3% for a half-year period).

You have a choice:

1. You can draw the diagram in terms of the ***payment period*** (3 months in this case) and determine the equivalent effective rate for this period; or
2. You can draw the diagram in terms of the ***interest period*** given in the problem (6 months in this case).

Following the first alternative (the payment period), the diagram would look like this:



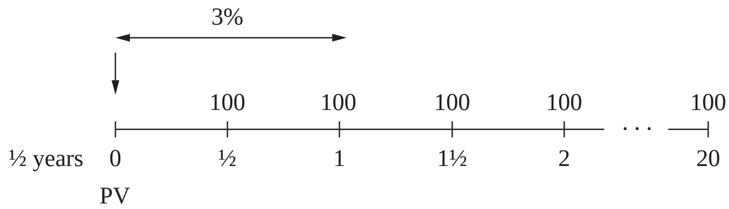
where $j = 1.03^{\frac{1}{2}} - 1 = 1.48892\%$.

(The thinking was: "I want the diagram to be in terms of *payment periods*. This will lead to the simplest form of the equation of value. Since payments are quarterly, the period in the diagram will be quarter-years and the payments will fall at time $1, 2, 3, \ldots$. Since there are 40 payments (quarterly payments for 10 years), the last payment is at time 40. The only thing left to do is to find the effective rate of interest for a quarter of a year—call it j—equivalent to the given nominal rate of 6% compounded semiannually. I can do this by accumulating 1 for one year (or for any period of time) at each rate: $1.03^2 = (1+j)^4$, from which $j = 1.03^{\frac{1}{2}} - 1$.")

The equation of value at time 0 is:

$$\boxed{\boldsymbol{PV = 100(v + v^2 + \ldots + v^{40})} \text{ at rate } \boldsymbol{j = 1.03^{\frac{1}{2}} - 1}}$$

Following the second alternative (the interest period), the diagram looks like this:



(The thinking was: "The effective rate of interest implied by 6% compounded semiannually is 3% for a half-year period, so I will label my diagram in terms of half-year interest periods. The payments are quarterly, or twice each interest period, so I will place the payments at time $\frac{1}{2}$, 1, $1\frac{1}{2}$, etc. This is a 10-year annuity, which consists of 20 interest periods, so the last payment is at time 20.")

The equation of value at time 0 is:

$$\boxed{PV = \boldsymbol{100(v^{\frac{1}{2}} + v^1 + v^{1\frac{1}{2}} + \ldots + v^{20})}}$$

where $v = 1.03^{-1}$.

Which method is better, using the payment period or using the interest period?

If you only want numerical results, using the payment period is generally the best approach. Interest functions that we will cover later in this manual are at their simplest when the interest period is the same as the payment period. You do have to calculate the effective rate for the payment period and to use it in any equation of value. The fact that this rate will usually not be a round number doesn't matter, since you can handle any rate with the calculator.

On the other hand, the answers to an exam question may be left in symbolic form, although this has happened very infrequently, if at all, in recent exams. If this is the case, chances are that the examiners will use symbols at the "original" effective rate (3% per half-year in the above problem). In that case, your thinking would have to be in terms of interest periods.

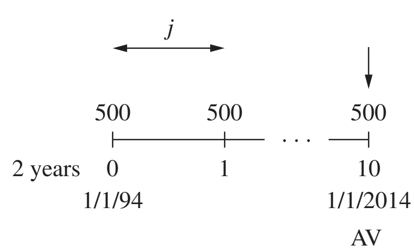
The upshot of this is that you really have to know both approaches and have facility in drawing diagrams and writing equations of value in terms of either the interest period or the payment period. This is covered in greater detail in Sections 4a and 4b of this manual.

Example 2.4

Draw a time diagram and write an equation of value for the following problem:

Deposits of \$500 are made on January 1 of even years only from 1994 to 2014 inclusive. Find the accumulated value on the date of the last deposit if the nominal rate of interest is 8% compounded quarterly.

Solution
Using the payment period:



where $1 + j = 1.02^8$; $j = 1.02^8 - 1 = 17.1659\%$.

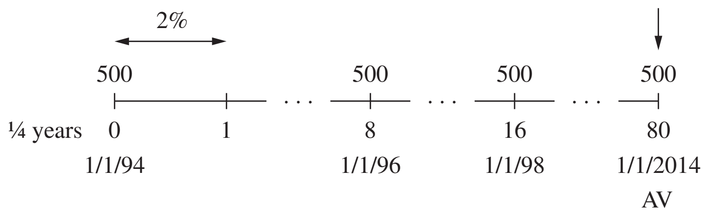
Equation of value:

$$\boxed{\boldsymbol{AV = 500[1 + (1 + j) + \ldots + (1 + j)^{10}]} \text{ at } \boldsymbol{j = 17.1659\%}}$$

(This can be expressed as $500\, s_{\overline{11|}}$ at rate j, using a symbol introduced in the next chapter.)

Note: You can number the time periods in the diagram any way you want to, as long as the payments are spaced correctly. For example, in the above diagram you could have used time 1 for 1/1/94 and time 11 for 1/1/2014. This would have placed the payments at times 1 to 11, inclusive. Verify that if you then choose time 11 (i.e., 1/1/2014) as the comparison date, you would get the same equation of value as the one above.

Using the interest period:



Equation of value:

$$\boxed{\boldsymbol{AV = 500(1 + 1.02^8 + 1.02^{16} + \cdots + 1.02^{80})}}$$

(In a later chapter, this could be expressed as $500\frac{s_{\overline{88}|}}{s_{\overline{8}|}}$ at $i = 2\%$.)

Note: Once again, you can number the time periods in the diagram any way you want to. For example, in the above diagram, you could have used time 8 for 1/1/94 and time 88 for 1/1/2014. Of course, this would have led to the same equation of value using time 88 as the comparison date.

2b Unknown Time and Unknown Interest Rate

Unknown Time

Problems involving unknown time can often be solved using the logarithm LN key or the ***TVM keys***. (See questions 5 and 6 in Calculator Notes #1.)

Stepping Stones

Example 2.5

500 accumulates to 1,500 in t years at an effective annual interest rate of 4%. Determine t.

Solution

$$(500)(1.04^t) = 1{,}500$$
$$1.04^t = 3$$

Using the LN key we have $t = \log 3/\log 1.04 = 1.098612/.039221 = \boxed{\mathbf{28.01}}$ years to two decimals.

Using the TVM keys, we enter: 4 I/Y 500 PV 1500 +/− FV CPT N for the same result.

Example 2.6

How long will it take money to double at an effective annual rate of interest of 5%?

Solution

$$1.05^t = 2$$

Using either logarithms or the TVM keys, you should get $t = \boxed{\mathbf{14.2067}}$ to 4 decimals. Note that this is question 5 in Calculator Notes #1.

In your work or readings, you may come across an approximation for t, sometimes called the ***Rule of 72***: The time it takes money to double at a specified interest rate is equal to 72 divided by the interest rate in percent. At 5%, this results in $72/5 = 14.4$ years. A better approximation is 69.3 divided by the interest rate, plus 0.35. At 5%, this gives 14.21, which is correct to two decimals. (These approximations are based on certain series expansions and the fact that the natural log of 2 is approximately 0.693.)

Example 2.7

Bill will receive two payments of 1,000 each, the first payment in t years and the second in $2t$ years. The present value of these payments is 1,200 at an effective annual rate of 4%. Determine t.

Solution

The equation of value as of time 0 is:

$$1000(v^t + v^{2t}) = 1200 \quad \text{at } 4\%$$
$$v^{2t} + v^t - 1.2 = 0$$

This is a quadratic in v^t. For convenience, we will substitute x for v^t:

$$x^2 + x - 1.2 = 0$$

Taking the positive root, you should get $x = .704159 = v^t$.

Using either the LN key or the TVM keys, you should get $t = \boxed{8.943}$ years to 3 decimals.

Example 2.8

Fund A grows at a force of interest $\delta_t = 1/(1+t)$. Fund B grows at a constant force of interest of 5%. Equal amounts are invested in each fund at time 0. Determine the time at which the excess of Fund A over Fund B is at a maximum.

Solution

In general, the accumulation function is equal to the base of the natural logarithms, e, raised to the power of the integral of the force of interest. Determine the accumulation functions for Fund A and Fund B, take the difference, take the derivative of this difference, and set it equal to 0 to determine a relative minimum or maximum:

$$\text{Accumulation function for Fund A} = e^{\int_0^t \frac{1}{1+r}dr} = e^{[\ln(1+r)]_0^t} = 1 + t$$
$$\text{Accumulation function for Fund B} = e^{.05t}$$

Excess of A over B (call it E(t)):

$$\mathrm{E}(t) = 1 + t - e^{.05t}$$

Take the derivative:

$$\mathrm{E}'(t) = 1 - .05e^{.05t}$$

Setting this equal to 0, we get $e^{.05t} = 20$ and $t = 20\ln 20 = \boxed{59.91}$ years to two decimals. (Note that the second derivative of E(t) is negative, which verifies that a relative maximum occurs at 59.91 years.)

Method of Equated Time

A type of problem involving unknown time is one where a series of payments is to be replaced by a single payment equal to the sum of the series. For example, suppose someone is scheduled to make three payments to you, as follows:

Time Due	Payment
1	5
3	1
10	15
Total Payments =	21

Now, suppose that in lieu of the scheduled payments, you and the other party agree that a ***single*** payment will be made to you equal to the ***sum*** of the scheduled payments. In this example, the single payment would be 21.

When should this single payment be made?

Clearly, this would depend on the interest rate we agree on, say, 5%. Let the time of the single payment be t. Then the following equation of value as of time 0 must be satisfied:

$$21v^t = 5v + v^3 + 15v^{10}$$

You can see that the solution for t involves taking logarithms of both sides, etc. This procedure will give the exact value of t. You can verify that if $i = 5\%$, $t = 7.12$ to 2 decimals.

The "***method of equated time***" is a method for determining an ***approximate*** value of t using simple arithmetic. The general idea is to find the arithmetic mean of the times at which the payments are due. But it wouldn't be a very good approximation if we just averaged the times at which each payment is due without regard to the ***amount*** due on each date. So to find t by the method of equated time, we compute the ***weighted*** average time. We will use the symbol $\bar{t}$ for the approximation:

$$\bar{t} = \frac{5 \times 1 + 1 \times 3 + 15 \times 10}{21} = \frac{158}{21} = 7.52$$

Note that the time at which each payment is due is weighted by the amount due: 5 is due at time 1 (5×1); 1 is due at time 3 (1×3); 15 is due at time 10 (15×10). Then the sum of these products (158) is divided by the sum of the weights (21) to give the weighted average time of 7.52.

This is certainly an approximation, since we didn't even use the interest rate in calculating it. In fact, the method of equated time gives the same answer regardless of the interest rate.

(Note that in this example, the method of equated time gives a result which is greater than the exact answer at 5%: $7.52 > 7.12$. This is not an accident. It can be shown that the approximation always exceeds the exact answer for a positive interest rate.)

Example 2.9

An annuity provides an infinite series of annual payments of d, $\frac{d^2}{2}$, $\frac{d^3}{3}$, ..., first payment one year from now, where d is the effective rate of discount. In lieu of these payments, a single payment equal to their sum is to be made at time t. Determine t using the method of equated time.

(A) 1 (B) i/δ (C) d/δ (D) $e^{\frac{i}{d}}$ (E) $e^{\frac{i}{d}} - 1$

Solution

This question tests you on a series that we covered in Section 1f, as well as testing you on the method of equated time.

To determine the approximate time using the method of equated time, we calculate the weighted average time. In this problem, there are an infinite number of payments, so the weighted average involves infinite series.

The sum of the products of the weights (payments) and the times is as follows:

$$\begin{aligned}
&(d)(1) + \left(\frac{d^2}{2}\right)(2) + \left(\frac{d^3}{3}\right)(3) + \cdots \\
&= d + d^2 + d^3 + \cdots \\
&= \frac{d}{1-d} \text{ (sum of an infinite geometric progression)}
\end{aligned}$$

You may recognize this as i or you can substitute $d = iv$ and $1 - d = v$ to obtain i.

To get the weighted average time, we have to divide this result (i) by the sum of the weights (payments), which is:

$$d + \frac{d^2}{2} + \frac{d^3}{3} + \cdots$$

If your memory has held out from Section 1f, you will recognize this as $-\ln(1-d)$, or δ. Therefore, the weighted average time is $\boxed{i/\delta}$. ANS. (B)

Example 2.10

Hannah is scheduled to receive the following payments at the end of the years shown:

End of Year	Scheduled Payment
4	100
10	200
12	300

Hannah agrees to accept a single payment equal to the sum of the scheduled payments at the time determined by the method of equated time. The effective annual interest rate is 6%. Let X be the excess of the present value of Hannah's scheduled payments over the present value of her single payment under the method of equated time. Determine X.

Solution
The time of Hannah's single payment under the method of equated time is:

$$\bar{t} = \frac{(4)(100) + (10)(200) + (12)(300)}{100 + 200 + 300} = \frac{6{,}000}{600} = 10$$

The present value of the single payment is $600v^{10} = 335.04$.

The present value of the scheduled payments is $100v^4 + 200v^{10} + 300v^{12} = 339.98$.

$$X = 339.98 - 335.04 = \boxed{\mathbf{4.94}}$$

Unknown Interest Rate

Problems involving an unknown rate of interest are among the most difficult to solve. This is because the unknown rate may be the root (or roots) of a polynomial (or even of a more complex function).

Example 2.11
At what rate of interest will a payment of 1 now and 2 in one year accumulate to 4 in 2 years?

Solution
The equation of value as of time 2 is:

$$(1+i)^2 + 2(1+i) = 4$$

For convenience, we will let $(1+i) = x$. We have:

$$x^2 + 2x - 4 = 0$$

Taking the positive root:

$$\begin{aligned} x &= \frac{-2 + \sqrt{2^2 - (4)(1)(-4)}}{2} \\ &= \frac{\sqrt{20} - 2}{2} = \frac{2\sqrt{5} - 2}{2} \\ &= \sqrt{5} - 1 = 1.236068 \end{aligned}$$

Thus, $x = 1 + i = 1.236068$ and $i = .236068$, or $\boxed{\mathbf{23.61\%}}$.

In this case, it was easy to solve for i, since it was the root of a quadratic. But with polynomials of higher degree, or for more complex functions, it could be very difficult to determine a numerical value for i without a financial calculator. (In fact, formulas do not exist for finding the roots of a polynomial of degree 5 or higher.) We might have to resort to an approximation or iterative technique, like interpolation, successive bisection, Newton-Raphson iteration, etc. But as we will see in later sections, a financial calculator such as the BA II Plus can be used to compute an unknown interest rate in many cases.

Example 2.12
At what effective annual rate of interest $i > 0$ is the present value of 100 due at time 5 plus the present value of 200 due at time 15 equal to the present value of 300 due at time 10?

Solution
The equation of value as of time 0 is:

$$100v^5 + 200v^{15} = 300v^{10}$$

Dividing both sides by $100v^5$ and rearranging:

$$2v^{10} - 3v^5 + 1 = 0$$

Let $x = v^5$:

$$2x^2 - 3x + 1 = 0$$

The left-hand side can be factored as follows:

$$(2x - 1)(x - 1) = 0$$

Taking the first root:

$$2x = 1,\ x = 0.5 = v^5 \text{ or } (1+i)^5 = 2$$

$$i = \boxed{\mathbf{14.87\%}} \quad \text{to 2 decimals}$$

(The other root is $x = 1 = v^5$, which means that $i = 0$, but we were asked to find $i > 0$.)

Summary of Concepts and Formulas in Sections 2a and 2b

1. We say that money has a time value, which means that \$1.00 paid today is equivalent to $(1+i)^t$ paid at time t (assuming that the effective rate is a constant i).

2. Using the principle of the time value of money, we can set up ***equations of value*** by evaluating all payments as of a common date called the ***comparison date***.

3. ***Time diagrams*** can be very helpful in solving problems. (See examples in Section 2a.)

4. The "***method of equated time***" is the weighted average time of a series of scheduled payments. It is a simple approximation to the exact time at which a single payment equal to the sum of the scheduled payments should be made in lieu of the scheduled payments.

Past Exam Questions on Sections 2a and 2b

Equation of Value and Time Value of Money

1. You are given two loans, with each loan to be repaid by a single payment in the future. Each payment includes both principal and interest.

   The first loan is repaid by a 3,000 payment at the end of four years. The interest is accrued at 10% per annum compounded semiannually.

   The second loan is repaid by a 4,000 payment at the end of five years. The interest is accrued at 8% per annum compounded semiannually.

   These two loans are to be consolidated. The consolidated loan is to be repaid by two equal installments of X, with interest at 12% per annum compounded semiannually. The first payment is due immediately and the second payment is due one year from now.

   Calculate X. [SOA 11/89 #1]

   (A) 2,459 (B) 2,485 (C) 2,504 (D) 2,521 (E) 2,537

2. Carl puts 10,000 into a bank account that pays an annual effective interest rate of 4% for ten years. If a withdrawal is made during the first five and one-half years, a penalty of 5% of the withdrawal amount is made.

   Carl withdraws K at the end of each of years 4, 5, 6 and 7. The balance in the account at the end of year 10 is 10,000.

   Calculate K. [SOA 5/89 #5]

   (A) 929 (B) 958 (C) 980 (D) 1,005 (E) 1,031

3. You are given the following data on three series of payments:

	Payment at end of year			
	6	**12**	**18**	**Accumulated value at end of year 18**
Series A	240	200	300	X
Series B	0	360	700	$X+100$
Series C	Y	600	0	X

   Assume interest is compounded annually.

   Calculate Y. [SOA 5/88 #4]

   (A) 93 (B) 99 (C) 102 (D) 107 (E) 111

4. A loan of 1000 is made at a nominal annual interest rate of 12% compounded quarterly. The loan is to be repaid with three payments: 400 at the end of the first year, 800 at the end of the fifth year, and the balance at the end of the tenth year.

   Calculate the amount of the final payment. [SOA 5/87 #1]

   (A) 587 (B) 658 (C) 737 (D) 777 (E) 812

5. Jones agrees to pay an amount of $2X$ at the end of 3 years and an amount of X at the end of 6 years. In return he will receive \$2,000 at the end of 4 years and \$3,000 at the end of 8 years.

   At an 8% effective annual interest rate, what is the size of Jones' second payment? [CAS 5/86 #4]

   (A) Less than \$1,250
   (B) At least \$1,250, but less than \$1,300
   (C) At least \$1,300, but less than \$1,350
   (D) At least \$1,350, but less than \$1,400
   (E) \$1,400 or more

6. At a certain interest rate the present value of the following two payment patterns are equal:

   (i) 200 at the end of 5 years plus 500 at the end of 10 years
   (ii) 400.94 at the end of 5 years

   At the same interest rate, 100 invested now plus 120 invested at the end of 5 years will accumulate to P at the end of 10 years.

   Calculate P. [SOA 5/86 #1]

   (A) 901 (B) 918 (C) 942 (D) 967 (E) 992

7. The XYZ Casualty Insurance Company has found that for a particular type of insurance policy it makes the following payments for insurance claims:

   (i) on 10% of the policies, XYZ Company pays $1,000 exactly one year after the effective date of the policy;
   (ii) on 3% of the policies, XYZ Company pays $10,000 exactly three years after the effective date of the policy;
   (iii) on the remaining policies, XYZ Company makes no payment for claims.

   In addition to the above payments, XYZ Company pays $20 for the expenses of administering the policy: $10 is paid on the effective date of the policy and the remaining $10 is paid six months after the effective date of the policy.

   The annual interest rate is 8%, compounded semiannually.

   The premium for this type of insurance policy is due six months after the effective date of the policy.

   If the present value of the premium is set equal to the present value of the claim payments and expenses, what is the premium? [CAS 11/82 #5]

   (A) Less than $355
   (B) At least $355 but less than $380
   (C) At least $380 but less than $415
   (D) At least $415 but less than $440
   (E) At least $440

Unknown Time and Unknown Interest Rate

8. David can receive one of the following two payment streams:

   (i) 100 at time 0, 200 at time n, and 300 at time $2n$
   (ii) 600 at time 10

   At an annual effective interest rate of i, the present values of the two streams are equal. Given $v^n = 0.75941$, determine i. [11/01 #24]

   (A) 3.5% (B) 4.0% (C) 4.5% (D) 5.0% (E) 5.5%

9. Joe deposits 10 today and another 30 in five years into a fund paying simple interest of 11% per year.

   Tina will make the same two deposits, but the 10 will be deposited n years from today and the 30 will be deposited $2n$ years from today. Tina's deposits earn an annual effective rate of 9.15%.

   At the end of 10 years, the accumulated amount of Tina's deposits equals the accumulated amount of Joe's deposits.

   Calculate n. [5/00 #1]

   (A) 2.0 (B) 2.3 (C) 2.6 (D) 2.9 (E) 3.2

10. An investment of $1 will double in 20 years at a force of interest δ.

    Determine the number of years required for an investment of $1 to triple at a nominal rate of interest, convertible 3 times per year, and which is numerically equivalent to δ. [CAS 5/98 #7]

    (A) Less than 31 years
    (B) At least 31 years, but less than 33 years
    (C) At least 33 years, but less than 35 years
    (D) At least 35 years, but less than 37 years
    (E) 37 years or more

Part II

Six Original Practice Exams

Practice Exam 1

Note to Students: These practice exams follow the format of the actual exams in October 2022 and subsequent: 30 questions in $2\frac{1}{2}$ hours. The actual exam will be in CBT format. A few of the questions will be pilot questions that will not be graded, but you will have no way of knowing which ones they are.

When you take these exams, stick to the time limit and simulate exam conditions.

Questions for Practice Exam 1

1. Which of the following is *not* correct with respect to an annual effective interest rate of $i = 10\%$?

   (A) $\delta = e^{0.10} - 1$

   (B) $i^{(2)} = 2 \times [(1.10)^{0.50} - 1]$

   (C) $\delta = \ln(1.10)$

   (D) $d = \frac{0.10}{1.10}$

   (E) $d^{(4)} = 4 \times [1 - (1.10^{-0.25})]$

2. You can receive one of the following two sets of cash flows. Under Option A, you will receive 10 annual payments of \$1,000, with the first payment to occur 4 years from now. Under Option B, you will receive X at the end of each year, forever, with the first payment to occur 1 year from now. The annual effective rate of interest is 8%. Which of the following equations should be solved to find the value of X such that you are indifferent between these two options?

   (A) $80a_{\overline{10}|}v^4 = X$ (B) $80a_{\overline{13}|}v^3 = X$ (C) $80a_{\overline{10}|}v^3 = X$ (D) $80a_{\overline{10}|}v^3(0.08) = X$ (E) $80a_{\overline{13}|}v^2 = X$

3. An annuity will pay you \$500 two years from now, and another \$1,000 four years from now. The present value of these two payments is \$1,200. Find the implied effective annual interest rate, i.

   (A) $i \le 4.5\%$ (B) $4.5\% < i \le 5.5\%$ (C) $5.5\% < i \le 6.5\%$ (D) $6.5\% < i \le 7.5\%$ (E) $7.5\% < i$

4. An investor took out a 30-year loan which he repays with annual payments of 1,500 at an annual effective interest rate of 4%. The payments are made at the end of the year. At the time of the 12^{th} payment, the investor pays an additional payment of 4,000 and wants to repay the remaining balance over 10 years. Calculate the revised annual payment.

   (A) 1,682 (B) 1,729 (C) 1,783 (D) 1,825 (E) 1,848

5. A 25-year loan is being paid off via level amortization payments made at the end of each quarter. The nominal annual interest rate is 12% convertible monthly. The amount of principal in the 29th payment is 1,860. Find the amount of principal in the 61st payment

   (A) 4,535 (B) 4,635 (C) 4,735 (D) 4,835 (E) 4,935

6. Suppose you are the actuary for an insurance company. Your company, in response to a policyholder claim involving physical injury, is responsible for making annual medical payments. The first payment will occur on January 1, 2008, and the final payment will occur on January 1, 2031. The first payment will be \$100,000; after that, the payments will increase annually for inflation, at a rate of 5% per year. The real interest rate is 3% per year. Find the present value of these future payments as of December 31, 2005.

   (A) 1,491,000 (B) 1,501,000 (C) 1,511,000 (D) 1,521,000 (E) 1,531,000

7. A company must pay the following liabilities at the end of the years shown:

End of Year	Liability
2	\$1,000
4	X
6	1,000

The company achieves Redington immunization by purchasing assets that have two cash inflows: \$733 at the end of one year and Y at the end of 5 years. The effective annual rate of interest is 10%. Determine Y.

(A) 1,789 (B) 1,934 (C) 2,152 (D) 2,201 (E) 2,376

8. An investment is expected to pay 2 one year from now, and 3 two years from now. Thereafter, payments are annual with each being g% greater than the previous payment. The effective annual interest rate is 8.5%, and the purchase price of this investment is 112.50. Find g.

(A) 5.6 (B) 5.7 (C) 5.8 (D) 5.9 (E) 6.0

9. At any moment t, a continuously-varying continuous 5-year annuity makes payments at the rate of t^2 per year at moment t. The force of interest is 6%. Which of the following represents a correct expression of the present value of this annuity?

(A) $\int_0^5 t^2 e^{0.06t} dt$

(B) $\int_0^5 t^2 e^{-0.06t} dt$

(C) $\int_0^5 t e^{-0.12t} dt$

(D) $\int_0^5 t^2 (1.06)^{-t} dt$

(E) None of (A), (B), (C), or (D) is a correct expression of the present value of the annuity.

10. A loan of 45,000 is being repaid with level annual payments of 3,200 for as long as necessary plus a final drop payment. All payments are made at the end of the year. The principal portion of the 9^{th} payment is 1.5 times the principal portion of the 2^{nd} payment. Calculate the drop payment.

(A) 1,495 (B) 1,521 (C) 1,546 (D) 1,584 (E) 1,597

11. A project requires an investment of 50,000 now (time 0), and will provide returns of X at the end of each of years 3 through 10. The effective annual rate of interest is 10%. The net present value of this project is 2,500. Find X.

(A) 11,300 (B) 11,500 (C) 11,700 (D) 11,900 (E) 12,100

12. Two growing perpetuities have the same yield rate. The first perpetuity—a perpetuity-immediate—has an initial payment of 500 one year from now, and each subsequent annual payment increases by 4%. This first perpetuity has a present value of 9,500. The second perpetuity—also a perpetuity-immediate—has an initial payment of 400 one year from now, and each subsequent annual payment increases by 20. Find the present value, P, of this second perpetuity.

(A) $P \leq 6{,}500$

(B) $6{,}500 < P \leq 6{,}600$

(C) $6{,}600 < P \leq 6{,}700$

(D) $6{,}700 < P \leq 6{,}800$

(E) $6{,}800 < P$

13. Jenna purchased an n-year \$1,000 par value bond at a discount to yield 4.2% convertible semiannually. The bond pays coupons at 3.6% convertible semiannually and has a redemption value of \$1,150. The purchase price is \$1,035. Calculate n.

(A) 6 (B) 8 (C) 12 (D) 16 (E) 24

14. A 10-year 200,000 loan is being paid off with level amortization payments at the end of each month. The effective annual interest rate is 15%. Find the amount of interest in the 56th monthly payment.

(A) 1,576 (B) 1,607 (C) 1,652 (D) 1,714 (E) 1,789

15. A 30-year \$300,000 loan involves level amortization payments at the end of each year. The effective annual interest rate is 9%. Let P be the ratio of total dollars of interest paid by the borrower divided by total aggregate payment dollars made by the borrower over the life of the loan. Find P.

(A) $P \leq 0.525$ (B) $0.525 < P \leq 0.575$ (C) $0.575 < P \leq 0.625$ (D) $0.625 < P \leq 0.675$ (E) $0.675 < P$

16. At the end of each year, for the next 19 years, you make deposits into an account, as follows:

$$\text{Deposit at end of year } t = 100t \quad \text{for } t = 1, 2, 3, \ldots, 10.$$
$$\text{Deposit at end of year } t = 1{,}000 - \{100(t-10)\} \quad \text{for } t = 11, 12, 13, \ldots, 19$$

The effective annual interest rate is 10%. Find the present value, at time $t = 0$, of this annuity.

(A) 4,053 (B) 4,103 (C) 4,153 (D) 4,203 (E) 4,253

17. An investment opportunity has the following characteristics: payments of \$10,000 will be made to you and invested into a fund at the beginning of each year, for the next 20 years. These payments will earn a 7% effective annual rate, and the interest payments (paid at the end of each year) will immediately be reinvested into a second account earning a 4% effective annual rate. Find the purchase price of this investment opportunity, given that it has an annual yield of 6% over the 20-year life of the investment.

(A) 92,000 (B) 102,000 (C) 112,000 (D) 122,000 (E) 132,000

18. A 30-year bond with par value 1,000 has annual coupons and sells for 1,300. The write down in the first year is 4.60. What is the yield-to-maturity for this bond?

(A) 4.73% (B) 4.89% (C) 4.98% (D) 5.15% (E) 5.27%

19. A \$7,600 loan is being repaid by level installments at the end of each year for 14 years. The annual effective rate of interest is 4% for the first 6 years and 5% thereafter. Which of the following formulas gives the amount of the level installment?

(A) $\dfrac{7,600}{a_{\overline{6}|4\%} + a_{\overline{8}|5\%}}$ (B) $\dfrac{7,600}{a_{\overline{14}|5\%} - a_{\overline{6}|4\%}}$ (C) $\dfrac{7,600}{a_{\overline{14}|4\%} - a_{\overline{8}|5\%}}$ (D) $\dfrac{7,600}{a_{\overline{6}|4\%}(1.05)^8 + a_{\overline{8}|5\%}}$ (E) $\dfrac{7,600}{a_{\overline{6}|4\%} + a_{\overline{8}|5\%}(1.04)^{-6}}$

20. A 20-year 100 par value bond with 8% semiannual coupons is purchased for 108.50. What is the book value of the bond just after the 13^{th} coupon is paid?

(A) 102.24 (B) 103.32 (C) 104.89 (D) 105.73 (E) 106.91

21. Yield rates to maturity for zero coupon bonds are currently quoted at 6% for one-year maturity, 7% for two-year maturity, and 7.5% for three-year maturity. Find the present value, two years from now, of a one-year 1000-par-value zero-coupon bond.

(A) 902 (B) 922 (C) 942 (D) 962 (E) 982

22. Determine the modified duration (or "volatility") of a growing perpetuity. The perpetuity will make annual payments, with the first payment being \$1 one year from now, and thereafter each subsequent payment will be \$1 greater than the preceding payment. Assume an annual effective interest rate of 8%.

(A) 12 (B) 16 (C) 20 (D) 24 (E) 28

23. You purchase a 7.5% annual coupon bond with a face value of 1,000 to yield a minimum interest rate of 8% effective. The bond is a callable corporate bond, with a call price of 1,050, and can be called by the issuing corporation after five years. The bond matures at par in 30 years. Immediately after the 12th coupon payment, the issuing corporation redeems the bond. Determine the effective annual yield you achieved on this twelve-year investment.

(A) 6.5% (B) 7.0% (C) 7.5% (D) 8.0% (E) 8.5%

24. A one-year zero-coupon bond has an annual yield of 6.25%. A two-year zero-coupon bond has an annual yield of 7.00%. A three-year zero-coupon bond has an annual yield of 7.50%. A three-year 12% annual coupon bond has a face value of \$1,000. Find the yield to maturity on this three-year 12% annual coupon bond.

(A) 6.6% (B) 7.0% (C) 7.4% (D) 7.8% (E) 8.2%

25. Bond A is an n-year 100 par value bond with 8% annual coupons and sells for 140.25. Bond B is an n-year 100 par value bond with 3% annual coupons and sells for 80.17. Both bonds have the same yield rate i. Determine i.

(A) 3.82% (B) 4.65% (C) 4.85% (D) 5.15% (E) 5.52%

26. A 30-year 1,000 par value bond pays 10% annual coupons. Using an interest rate of 12%, find the Macaulay duration of this bond.

(A) 9.2 (B) 10.2 (C) 11.2 (D) 12.2 (E) 13.2

27. An insurer must pay 3,000 and 4,000 at the ends of years 1 and 2, respectively. The only investments available to the company are a one-year zero-coupon bond (with a par value of 1,000 and an effective annual yield of 5%), and a two-year 8% annual coupon bond (with a par value of 1,000 and an effective annual yield of 6%). Which of the following is closest to the cost to the company today to match its liabilities exactly?

(A) 6,014 (B) 6,114 (C) 6,214 (D) 6,314 (E) 6,414

28. Sue decided to purchase a 20-year annuity that pays \$900 at the end of the first year, \$915 at the end of the second year, and for each year thereafter the payment increases by \$15. Which of the following formulas gives the price of this annuity?

(A) $900 + 15(Ia)_{\overline{19}|}$ (B) $885 + 15(Ia)_{\overline{20}|}$ (C) $900a_{\overline{20}|} + 15(Ia)_{\overline{20}|}$ (D) $900a_{\overline{20}|} + 15(Ia)_{\overline{19}|}$ (E) $885a_{\overline{20}|} + 15(Ia)_{\overline{20}|}$

29. Christine deposits \$100 into an account which earns interest at an effective annual rate of discount of d. At the same time, Douglas deposits \$100 into a separate account earning interest at a force of interest of $\delta_t = 0.001t^2$. After 10 years, both accounts have the same value. Find d.

(A) 3.3% (B) 3.6% (C) 3.9% (D) 4.2% (E) 4.5%

30. You are given the following information about two annual-coupon bonds, each with a face and redemption value of \$ 1,000, and each 3 years in length:

- Bond A: A 3-year 6% annual coupon bond with a price of \$955.57.
- Bond B: A 3-year 8% annual coupon bond with a price of \$1,008.38.

Using this data, find the annual yield on a 3-year zero-coupon bond.

(A) 6.40% (B) 6.95% (C) 7.30% (D) 7.85% (E) 8.40%

Solutions to Practice Exam 1

1. All of the formulas except the first (answer (A)) are valid equivalencies when the effective rate of interest is 10%. The correct relationship between the effective rate and the force of interest is $e^\delta = 1 + i$ or $i = e^\delta - 1$ or $\delta = \ln(1 + i)$. ANS. (A)

2. "Indifference" between two alternatives means that a person considers the present values of the two options to be equal. Setting up this equivalency relationship:

$$\boxed{\mathbf{1{,}000 \cdot a_{\overline{10}|.08} \cdot v^3 = \frac{X}{0.08}}}$$

which is equivalent to answer (C). The three-year present value factor on the left-hand side is necessary because the first payment is four years away, and the annuity-immediate formula provides a PV one year prior to the first payment (leaving three more years of discounting to invoke). ANS. (C)

3. Set up the present value formula. The key is to recognize this as a quadratic in v^2:

$$\begin{aligned} 1{,}200 &= 500v^2 + 1{,}000v^4 \\ 10(v^2)^2 + 5v^2 - 12 &= 0 \\ v^2 &= \frac{-5 \pm \sqrt{25 + 480}}{20} = 0.873610 \\ v &= 0.934671 \\ i &= \boxed{\mathbf{0.0699}} \qquad \text{ANS. (D)} \end{aligned}$$

4. The outstanding balance at time 12 prior to the additional payment is:

$$B_{12} = 1{,}500a_{\overline{18}|.04} = 18{,}988.95.$$

After the additional payment, the outstanding balance is 14,988.95.

To pay this remaining balance in 10 years, the revised annual payment is such that:

$$Pa_{\overline{10}|.04} = 14{,}988.95 \quad \text{that gives } P = \boxed{\mathbf{1{,}848.00}} \qquad \text{ANS. (E)}$$

5. The key in this problem is to use the $(1 + i)$ multiplicative factor relationship between the principal components of sequential amortization payments. This is a consequence of the formula $P_t = R \cdot v^{n-t+1}$. Thus, once the appropriate interest rate is determined, the answer can be found quickly:

$$\begin{aligned} j &= (1.01)^3 - 1 = 0.030301/qtr \\ P_{61} &= P_{29} \cdot (1 + j)^{32} = \boxed{\mathbf{4{,}834.65}} \qquad \text{ANS. (D)} \end{aligned}$$

6. This is an application of a geometrically-growing annuity present value function. It can be done using either real payments and interest rates, or nominal payments and rates. Using the latter approach:

$$\begin{aligned} i_{\text{NOM}} &= (1.05 \times 1.03) - 1 = 0.0815 \\ PV &= v_i \cdot 100{,}000 \cdot \left(\frac{1 - \left(\frac{1.05}{1.0815}\right)^{24}}{0.0815 - 0.05} \right) = \boxed{\mathbf{1{,}491{,}363}} \qquad \text{ANS. (A)} \end{aligned}$$

7. The first condition of Redington immunization is $P_A = P_L$, where P_A is the PV of the assets and P_L is the PV of the liabilities:

$$(1) \quad 733v + Yv^5 = 1000v^2 + Xv^4 + 1000v^6$$

Dividing (1) by v:

$$(2) \quad 733 + Yv^4 = 1000v + Xv^3 + 1000v^5$$

The second condition is $P'_A = P'_L$:

$$(3) \quad -733v^2 - 5Yv^6 = -2000v^3 - 4Xv^5 - 6000v^7$$

Dividing (3) by $-v^2$:

$$(4) \quad 733 + 5Yv^4 = 2000v + 4Xv^3 + 6000v^5$$

