ACTEX

LEARN TODAY. LEAD TOMORROW.

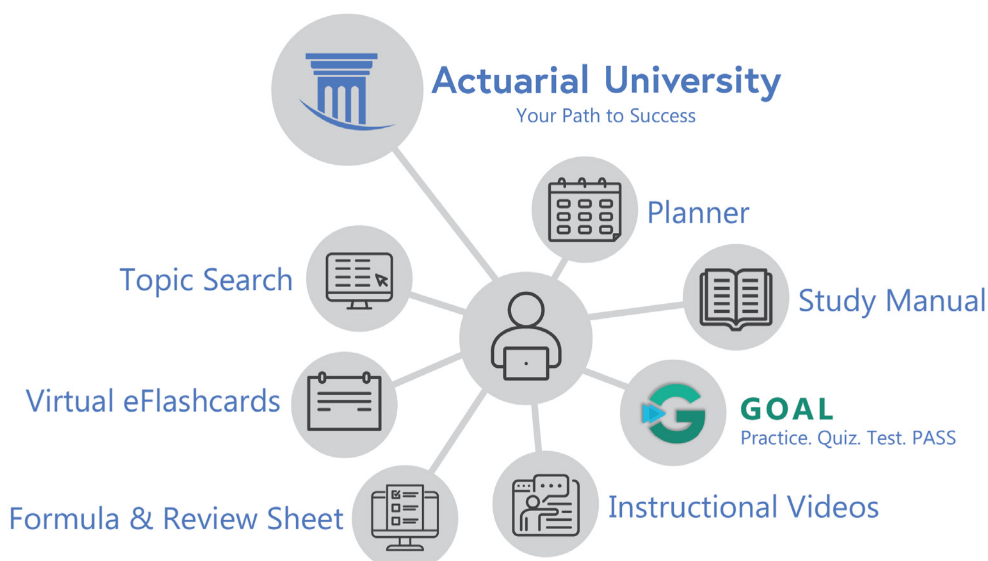
EXAM FM

Study Manual for SOA Exam FM

1st Edition, 2nd Printing

by
John B. Dinius, FSA
Matthew J. Hassett, Ph.D.
Michael I. Ratliff, Ph.D., ASA
Toni Coombs Garcia
Amy C. Steeby, MBA, MEd

Your Integrated Study Manual Program Includes:



ACTEX

Study Manual for SOA Exam FM

1st Edition, 2nd Printing

by
John B. Dinius, FSA
Matthew J. Hassett, Ph.D.
Michael I. Ratliff, Ph.D., ASA
Toni Coombs Garcia
Amy C. Steeby, MBA, MEd

ACTEX Learning

Learn Today. Lead Tomorrow.

Actuarial & Financial Risk Resource Materials

Since 1972

Printed in the United States of America.

Welcome to Actuarial University

Actuarial University is a reimagined platform built around a more simplified way to study. It combines all the products you use to study into one interactive learning center.

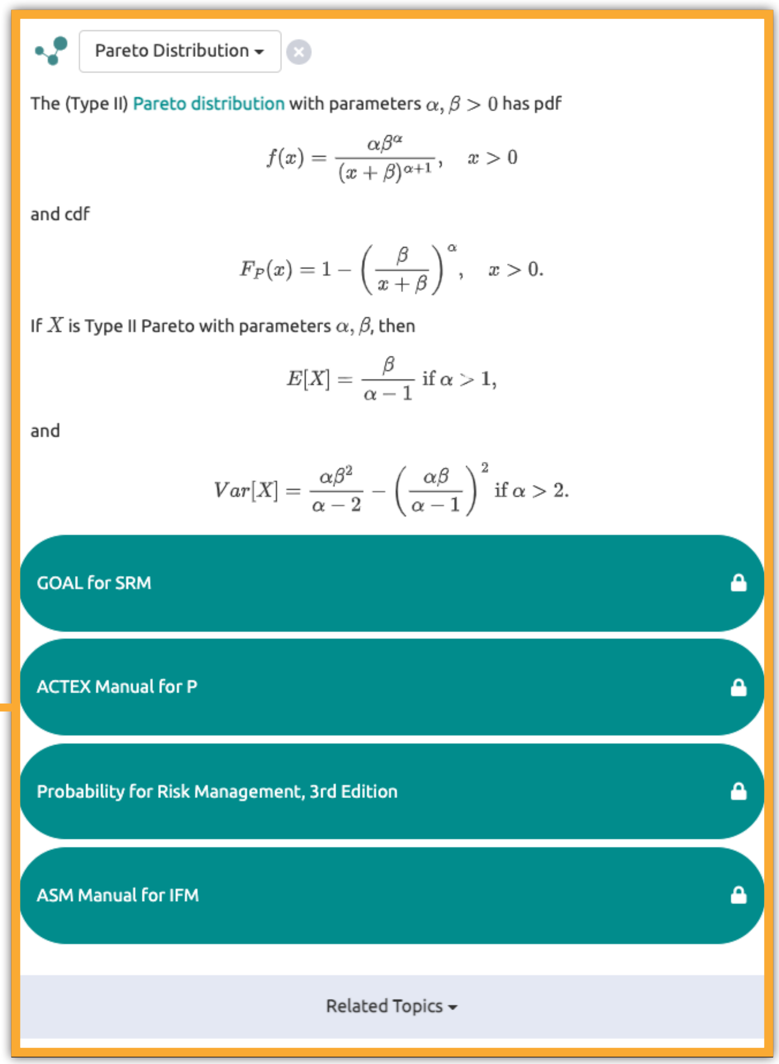
You can find integrated topics using this network icon.
When this icon appears, it will be next to an important topic in the manual! **Click the link** in your digital manual, or **search the underlined topic** in your print manual.

1. Login to: **www.actuarialuniversity.com**

2. Locate the **Topic Search** on your exam dashboard, and enter the word or phrase into the search field, selecting the best match.

3. A topic "**Hub**" will display a list of integrated products that offer more ways to study the material.

4. Here is an example of the topic **Pareto Distribution**:



Within the **Hub**, there will be unlocked and locked products.

Unlocked Products are the products that you own.

GOAL for P →

Locked Products are products that you do not own, and are available for purchase.

Probability for Risk Management, 3rd Edition

Many of Actuarial University's features are already unlocked with your study program, including:

GOAL Practice Tool	**Instructional Videos***
Virtual Flashcards	**Formula & Review Sheet**

GOAL: Guided Online Actuarial Learning

Your Adaptable, Customizable, Online Actuarial Exam Prep Tool.

GOAL is an **eLearning test preparation tool** for students to practice skills learned in class or from independent study. The online platform offers a massive database of **SOA & CAS exam-style problems with detailed solutions**.

GOAL offers **instructor support** as well as **performance reporting and tracking** to monitor your progress. Don't forget to check out **GOAL Score** to see how ready you are to sit for your exam!

Practice. Quiz. Test. Pass!

- 10,000+ Exam-style problems with detailed solutions!
- Adaptive Quizzes
- 3 Learning Modes
- 3 Difficulty Modes

How GOAL Works

**Available standalone, or included with the Study Manual Program Video Bundle*

Coming Soon for FAM, ASTAM & ALTAM!

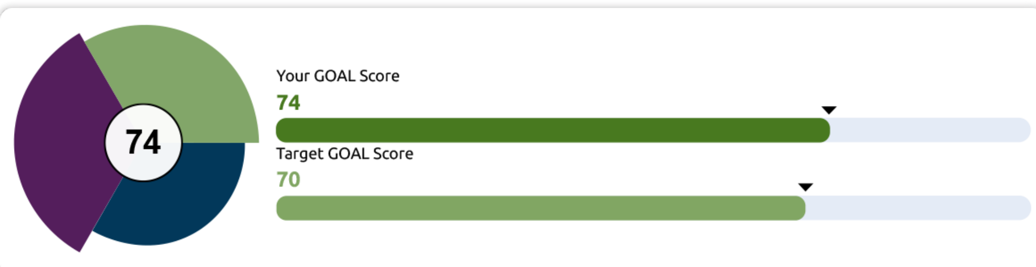
GOAL

Track your exam readiness with GOAL Score!

Available now for P, FM, IFM, STAM, SRM, MAS-I & MAS-II

GOAL Score tracks your performance through GOAL Practice Sessions, Quizzes, and Exams, resulting in an aggregate, weighted score that gauges your exam preparedness.

By measuring both **your performance**, and **the consistency of your performance**, **GOAL Score** produces a reliable number that will give you confidence in your preparation before you sit for your exam.



If your GOAL Score is a 70 or above, you are well-prepared to sit for your exam!

62	74	78
General Probability	Random Variables with Univariate Probability Distributions	Random Variables with Univariate Probability Distributions
Weighting: 13.4%	Weighting: 43.3%	Weighting: 43.3%

See key areas where you can improve.

General Probability — Set functions including set notation and basic elements of probability

Difficulty	Mean	Proportion
Core	58	92 / 304
Advanced	60	169 / 304
Mastery	78	43 / 304

Detailed performance tracking.

GOAL Session Activity Summary — Filter: All Types, All Statuses

Created	Last Accessed	Completed	Mode	Categories	Questions	Status	Status
05/24/2022 21:57:43	05/24/2022 21:57:43		Quiz	Continuous P...	25	New	Resume
05/24/2022 10:34:05	05/24/2022 14:57:49	05/24/2022 14:57:49	Practice Session	Addition and ...	80	Complete	Review
05/21/2022 19:32:50	05/23/2022 20:02:00		Simulated Exam	Exam 2	30	Reviewing	Complete
05/17/2022 15:19:19	05/17/2022 15:46:03	05/23/2022 14:15:29	Simulated Exam	Exam 6	30	Complete	Review
05/14/2022 11:26:59	05/14/2020 12:02:36	05/23/2022 11:57:47	Quiz	Conditional D...	20	Complete	Review

Quickly return to previous sessions.

Preface

ACTEX first published a study manual for the Society of Actuaries' Exam FM ("Financial Mathematics") in 2004. That manual was prepared by lead author Matthew Hassett, assisted by Michael Ratliff, Toni Coombs Garcia, and Amy Steeby. The manual has been regularly updated and expanded to keep pace with changes in the SOA's Exam FM syllabus. This latest edition of the ACTEX Study Manual for Exam FM, edited by lead author John Dinius, has been updated to reflect the current SOA syllabus (which is effective for exams administered in October 2022 and later). It provides over 1,000 examples, exercises, and problems to help you prepare for Exam FM.

This manual contains 9 learning modules, the first 7 of which cover all the material required for Exam FM. Modules 1 and 2 provide in-depth explanations and methodologies for the basic concepts of interest theory (the Time Value of Money and Annuities). Modules 3 and 4 apply these concepts to practical situations (Loans and Bonds). Modules 5, 6, and 7 cover more advanced topics, including Internal Rate of Return, the Term Structure of Interest Rates, and Asset-Liability Management. The last two modules (8 and 9) are no longer part of the Exam FM syllabus, but their material may be of interest to students who want to expand their knowledge of interest theory.

The manual provides three "midterm exams" that allow you to test your knowledge of the material you just learned. These exams are located after Modules 2, 4, and 7. At the end of the manual there are 14 full-length practice exams of 30 problems each. All of the problems in these midterm and practice exams are original and are not available anywhere else. They are intended to provide a realistic exam-taking experience to help you complete your preparation for Exam FM.

The following pages provide recommendations on how to prepare for actuarial exams and suggestions for using this manual most effectively.

> Any errors in this manual that are identified after it is published will be posted on the ACTEX website (www.actexmadriver.com) under the "Errata" link. We suggest that you access that link (selecting ACTEX Study Materials and Exam FM) and look for "ACTEX Exam FM Study Manual, 2023 Edition." If errata for this manual have been posted, you should make the appropriate changes in your copy of the manual.
>
> If you find a possible error in this manual, please let us know about it. You can click on the "Feedback" link on the left side of the ACTEX homepage (www.actexmadriver.com) and describe the issue. We will review and respond to all comments. Any confirmed errata will be posted on the ACTEX website under the "Errata" link.

On Passing Exams

How to Learn Actuarial Mathematics and Pass Exams

On the next page you will find a list of study tips for learning the material in the Exam FM syllabus and passing Exam FM. But first it is important to state the basic learning philosophy that we are using in this guide:

You must master the basics before you proceed to the more difficult problems.

Think about your basic calculus course. There were some very challenging applications in which you used derivatives to solve hard max-min problems.

It is important to learn how to solve these hard problems, but if you did not have the basic skills of taking derivatives and manipulating algebraic expressions, you could not do the more advanced problems. Thus every calculus book has you practice derivative skills before presenting the tougher sections on applied problems.

You should approach interest theory the same way. The first 2 or 3 modules give you the basic tools you will need to solve the problems in the later modules. Learn these concepts and methods (and the related formulas) very well, as you will need them in each of the remaining modules.

This guide is designed to progress from simpler problems to harder ones.

In each module we start with the basic concepts and simple examples, and then progress to more difficult material so that you will be prepared to attack actual exam problems by the end of the module.

The same philosophy is used in our practice exams at the end of this manual. The first few practice exams have simpler problems, and the problems become more difficult as you progress through the practice exams.

A good strategy when taking an exam is to answer all of the easier problems before you tackle the harder ones.

An exam is scored in percentage terms, and a multiple choice exam like Exam FM will have a mix of problems at different difficulty levels.

If an exam has ten problems and three are very hard, getting the right answers to only the three hard problems and missing the others gets you a score of 30%. This could happen if the hardest problems are the first ones on the exam and you attempt them first and never get to the easy problems.

A useful exam strategy is to go through the exam and quickly solve all the more basic problems before spending extra time on the hard ones. Strive to answer all of the easy problems correctly.

Study Tips

1) Develop a schedule so that you will complete your studying in time for the exam. Divide your schedule into time for each module, plus time at the end to review and to solve practice problems. Your schedule will depend on how much time you have before the exam, but a reasonable approach might be to complete one module per week.
2) If possible, join a study group of your peers who are studying for Exam FM.
3) For each module:
    a) Read the module in the FM manual (and the associated SOA study note, if any).
    b) As you read through the examples in the text, make sure that you can correctly compute the answers.
    c) Summarize each concept you learn in the manual's margins (if you have the print version) or in a notebook.
    d) Understand the main idea of each concept and be able to summarize it in your own words. Imagine that you are trying to teach someone else this concept.
    e) While reading, create flash cards for the formulas to facilitate memorization.
    f) Learn the calculator skills thoroughly and know *all* of the calculator's functions.
    g) Do the Basic Review Problems and review your solutions.
    h) Do the Sample Exam Problems and review your solutions.
        i) If you have been stuck on a problem for more than 20 minutes, it is OK to refer to the solutions. Just make sure that when you are finished with the problem, you can recite the concept that you missed and summarize it in your own words. If you get stuck on a problem, think about what principles were used in this question and see if you could write a different problem with similar structure (as if you were the exam writer).
        ii) Mark each sample exam problem as an Easy, Medium, or Hard problem.
    i) Do the Supplemental Exercises and review your solutions.
4) After learning the material in each module, it is a good idea to go back to previous modules and do a quick half-hour or 1-hour review, so that information isn't forgotten.
5) Go back and redo the sample exam problems that you marked as Medium or Hard when you worked through them the first time.
6) At the end of Modules 2, 4, and 7, we have included practice exams that are like midterms. Taking these tests will help you consolidate your knowledge.

7) After learning the material in all of the modules and taking the midterms, go to the practice exams.

   a) The first 6 practice exams are relatively straightforward to enable you to review the basics of each topic.

   b) The next 5 practice exams introduce more difficult questions in order to replicate the actual exam experience.

   c) The last 3 practice exams include especially challenging problems to test your understanding of the material and your ability to apply solution techniques.

   Please keep in mind that the actual exam questions are confidential, and there is no guarantee that the questions you encounter on Exam FM will look exactly like the problems in this manual.

Contents

Preface **ix**
On Passing Exams x
Study Tips xi

Introduction **xvii**

Module 1 Interest Rates and the Time Value of Money **1**
Section 1.1 Time Value of Money 1
Section 1.2 Present Value and Future Value 2
Section 1.3 Functions of Investment Growth 5
Section 1.4 Effective Rate of Interest 8
Section 1.5 Nominal Rate of Interest 10
Section 1.6 Rate of Discount 13
Section 1.7 Present Value Factor 15
Section 1.8 Nominal Rate of Discount 16
Section 1.9 Continuous Compounding and the Force of Interest 18
Section 1.10 Quoted Rates for Treasury Bills 23
Section 1.11 Relating Discount, Force of Interest, and Interest Rate 28
Section 1.12 Solving for PV, FV, n, and i 29
Section 1.13 The Rule of 72 37
Section 1.14 Formula Sheet 39
Section 1.15 Basic Review Problems 40
Section 1.16 Basic Review Problem Solutions 41
Section 1.17 Sample Exam Problems 45
Section 1.18 Sample Exam Problem Solutions 49
Section 1.19 Supplemental Exercises 56
Section 1.20 Supplemental Exercise Solutions 57

Module 2 Annuities **61**
Section 2.1 Introduction to Annuities 61
Section 2.2 Annuity-Immediate Calculations 62
Section 2.3 Perpetuities 66
Section 2.4 Annuity-Due Calculations 67
Section 2.5 Continuously Payable Annuities 70
Section 2.6 Basic Annuity Problems for Calculator Practice 73
Section 2.7 Annuities with Varying Payments 77
Section 2.8 Increasing Annuities with Terms in Arithmetic Progression 78
Section 2.9 Decreasing Annuities with Terms in Arithmetic Progression 81
Section 2.10 A Single Formula for Annuities with Terms in Arithmetic Progression . . . 83
Section 2.11 Annuities with Terms in Geometric Progression 86
Section 2.12 Equations of Value and Loan Payments 92
Section 2.13 Deferred Annuities 94
Section 2.14 Annuities With More Complex Payment Patterns 96
Section 2.15 Annuities with Payments More Frequent than Annual 100
Section 2.16 Annuities with Payments Less Frequent than Annual 104

Section 2.17 Payment Periods that Don't Match the Interest Conversion Periods 108
Section 2.18 Continuously Payable Annuities With Continuously Varying Payments . . . 114
Section 2.19 Reinvestment Problems 119
Section 2.20 Inflation 121
Section 2.21 Formula Sheet 123
Section 2.22 Basic Review Problems 127
Section 2.23 Basic Review Problem Solutions 129
Section 2.24 Sample Exam Problems 135
Section 2.25 Sample Exam Problem Solutions 141
Section 2.26 Supplemental Exercises 153
Section 2.27 Supplemental Exercise Solutions 155

Midterm 1 **161**

Module 3 Loan Repayment **171**
Section 3.1 The Amortization Method of Loan Repayment 171
Section 3.2 Calculating the Loan Balance 173
Section 3.3 Loans With Varying Payments 179
Section 3.4 Formulas for Level-Payment Loan Amortization 182
Section 3.5 Monthly-Payment Loans 184
Section 3.6 An Example With Level Payment of Principal 187
Section 3.7 Capitalization of Interest and Negative Amortization 189
Section 3.8 Formula Sheet 191
Section 3.9 Basic Review Problems 192
Section 3.10 Basic Review Problem Solutions 193
Section 3.11 Sample Exam Problems 196
Section 3.12 Sample Exam Problem Solutions 198
Section 3.13 Supplemental Exercises 201
Section 3.14 Supplemental Exercise Solutions 202

Module 4 Bonds **205**
Section 4.1 Introduction to Bonds 205
Section 4.2 Amortization of Premium or Discount 210
Section 4.3 Callable Bonds 214
Section 4.4 Pricing Bonds Between Payment Dates 218
Section 4.5 Formula Sheet 219
Section 4.6 Basic Review Problems 220
Section 4.7 Basic Review Problem Solutions 221
Section 4.8 Sample Exam Problems 222
Section 4.9 Sample Exam Problem Solutions 226
Section 4.10 Supplemental Exercises 231
Section 4.11 Supplemental Exercise Solutions 232

Midterm 2 **235**

Module 5 Yield Rate of an Investment **247**
Section 5.1 Internal Rate of Return 247
Section 5.2 Net Present Value 254
Section 5.3 Time-Weighted and Dollar-Weighted Rates of Return 256
Section 5.4 Formula Sheet 261

Section 5.5 Basic Review Problems . . . 262
Section 5.6 Basic Review Problem Solutions . . . 263
Section 5.7 Sample Exam Problems . . . 265
Section 5.8 Sample Exam Problem Solutions . . . 268
Section 5.9 Supplemental Exercises . . . 272
Section 5.10 Supplemental Exercise Solutions . . . 274

Module 6 The Term Structure of Interest Rates 277
Section 6.1 Spot Rates and the Yield Curve . . . 277
Section 6.2 Forward Rates . . . 284
Section 6.3 Formula Sheet . . . 288
Section 6.4 Basic Review Problems . . . 289
Section 6.5 Basic Review Problem Solutions . . . 290
Section 6.6 Sample Exam Problems . . . 292
Section 6.7 Sample Exam Problem Solutions . . . 294
Section 6.8 Supplemental Exercises . . . 296
Section 6.9 Supplement Exercise Solutions . . . 297

Module 7 Asset-Liability Management 301
Section 7.1 Basic Asset-Liability Management: Matching Asset & Liability Cash Flows . 301
Section 7.2 Duration . . . 303
Section 7.3 Modified Duration . . . 306
Section 7.4 Helpful Formulas for Duration Calculations . . . 309
Section 7.5 Using Duration to Approximate Change in Price . . . 311
Section 7.6 Approximations using Duration and Convexity . . . 320
Section 7.7 The Duration of a Portfolio . . . 324
Section 7.8 Immunization . . . 327
Section 7.9 Stocks and Other Investments . . . 333
Section 7.10 Formula Sheet . . . 338
Section 7.11 Basic Review Problems . . . 341
Section 7.12 Basic Review Problem Solutions . . . 343
Section 7.13 Sample Exam Problems . . . 347
Section 7.14 Sample Exam Problem Solutions . . . 352
Section 7.15 Supplemental Exercises . . . 359
Section 7.16 Supplemental Exercise Solutions . . . 361

Midterm 3 365

The last two modules (8 and 9) are no longer part of the Exam FM syllabus

Module 8 Determinants of Interest Rates 377
Section 8.1 Background . . . 377
Section 8.2 Components of the Interest Rate . . . 379
Section 8.3 Retail Savings and Lending Interest Rates . . . 390
Section 8.4 Bond Yields . . . 394
Section 8.5 The Role of Central Banks . . . 404
Section 8.6 Formula Sheet . . . 407
Section 8.7 Basic Review Problems . . . 408
Section 8.8 Basic Review Problem Solutions . . . 410
Section 8.9 Sample Exam Problems . . . 413

Section 8.10 Sample Exam Problem Solutions 415
Section 8.11 Supplemental Exercises 416
Section 8.12 Supplemental Exercise Solutions 419

Module 9 Interest Rate Swaps 427
Section 9.1 Introduction to Derivative Securities 427
Section 9.2 Variable-Rate Loans 428
Section 9.3 Example of an Interest Rate Swap 430
Section 9.4 Interest Rate Swap Terminology 433
Section 9.5 Calculating the Swap Rate 437
Section 9.6 Simplified Formulas for the Swap Rate 443
Section 9.7 Market Value of a Swap 447
Section 9.8 Formula Sheet 451
Section 9.9 Basic Review Problems 453
Section 9.10 Basic Review Problem Solutions 454
Section 9.11 Sample Exam Problems 457
Section 9.12 Sample Exam Problem Solutions 462
Section 9.13 Supplemental Exercises 466
Section 9.14 Supplemental Exercise Solutions 468

About the Practice Exams 471

Practice Exam 1 473

Practice Exam 2 489

Practice Exam 3 507

Practice Exam 4 527

Practice Exam 5 545

Practice Exam 6 565

Practice Exam 7 585

Practice Exam 8 605

Practice Exam 9 625

Practice Exam 10 647

Practice Exam 11 669

Practice Exam 12 693

Practice Exam 13 717

Practice Exam 14 739

Appendix The Texas Instruments BA II Plus Calculator 763

Index 778

Introduction

As you begin your preparation for the Society of Actuaries' Exam FM, you should be aware that studying Financial Mathematics (or "interest theory," as I like to call it) is not a matter of learning mathematics. Instead, financial mathematics involves *applying* mathematics to situations that involve financial transactions. This will require you to learn a new language, the language of the financial world, and then to apply your *existing* math skills to solve problems that are presented in this new language. It is important that you spend adequate time to fully understand the meanings of all the terms that will be introduced in this manual. Nearly all of the problems on Exam FM will be word problems (rather than just formulas), and it is very difficult to solve these problems unless you understand the language that is being used.

In this manual, we assume that you have a solid working knowledge of differential and integral calculus and some familiarity with probability. We also assume that you have an excellent knowledge of algebraic methods. Depending on what mathematics courses you have taken (and how recently), you may need to review these topics in order to understand some of the material and work the problems in this manual.

Throughout this manual, a large number of the examples and practice problems are solved using the Texas Instruments BA II Plus calculator, which is the financial calculator approved for use on Exam FM. It is essential for you to have a BA II Plus calculator in order to understand the solutions presented here, and also to solve the problems on the actual exam. This calculator is available in a standard model, and also as the "BA II Plus *Professional*." The Professional model, which is somewhat more expensive than the standard model, is a bit easier to work with, which could be important when taking a timed exam.

At the end of the manual there is an appendix with information about the BA II Plus. This appendix is included to help you learn the calculator's functions and adjust its settings so that you will be able to solve problems more quickly. Very importantly, the appendix explains that your calculator will be "reset" by the exam staff when you check in to take Exam FM, and provides instructions for returning the calculator to the settings you prefer. Reading this material will help you avoid having calculator difficulties on the day of your exam.

Over the years, most actuarial students have found that the best way to prepare for Exam FM is to work a very large number of problems (hundreds and hundreds of problems). There are many examples, exercises, problems, and practice exams included in this manual. Many more problems can be found on the Society of Actuaries website (www.soa.org) or by searching the Web. You should plan to spend a significant proportion of your study time working problems and reviewing the solutions that are provided in this manual and on the websites.

Financial mathematics is an integral part of an actuary's skill set, and you can expect to apply interest theory regularly throughout your career. Mastering the topics covered in this manual will provide you a valuable tool for understanding financial and economic matters both on and off the job.

Best of luck to you in learning Financial Mathematics and passing Exam FM!

John Dinius
November 2022

Module 2

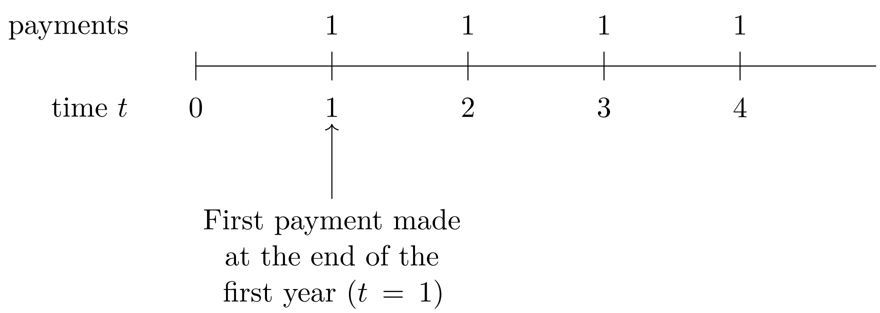
Annuities

Section 2.1 Introduction to Annuities

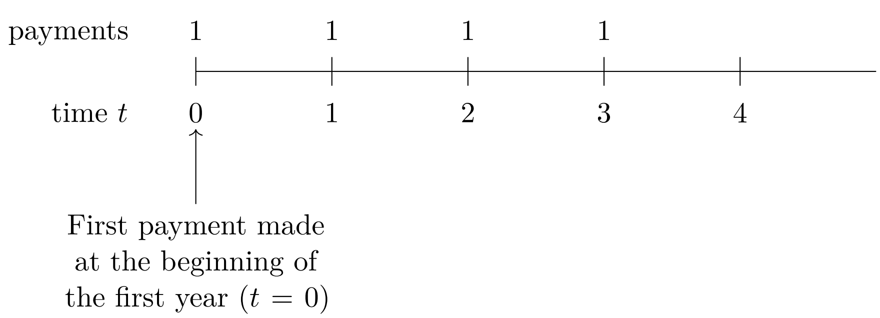
Many financial obligations require a series of periodic payments. Mortgage and car loan payments are usually made monthly. A retiree's pension plan typically pays a set amount at the beginning of every month. Premiums for an insurance policy might be paid monthly, quarterly, semi-annually, or annually. Series of regular payments such as these are called **annuities**. If the payments continue for a fixed **period** (e.g., 10 years), the annuity is called an **annuity-certain**. If the payment period is *not* fixed (e.g., a pension plan that makes monthly payments only as long as the retiree survives), it is a *contingent* annuity. (In the case of the pension, it is a "*life*-contingent annuity," or simply a "life annuity.") Exam FM deals with annuities-certain. Contingent annuities are covered in later exams.

A **unit annuity** is one for which each periodic payment is 1. Annuity payments may be made at the beginning or the end of each time period. If an annuity's payments occur at the end of each period, it is called an **annuity-immediate**. If the payments are made at the beginning of each period, it is an **annuity-due**. The diagrams below illustrate the payment patterns for unit annuities with four annual payments.

Annuity-Immediate



Annuity-Due



The preceding diagrams are called **timelines**. You will find them to be very useful in visualizing payment patterns and solving annuity problems.

Geometric Series

To find the present value or future value of an annuity, we will need to use the formula for the sum of a **geometric series**. Geometric series are very important for Exam FM. Consider the geometric series with n terms where the first term is 1 and the common ratio is r:

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r} = \frac{r^n - 1}{r-1}, \; r \neq 1 \tag{2.1}$$

In order for this equality to be valid, n must be an integer. The geometric series is not properly defined if n is not an integer.

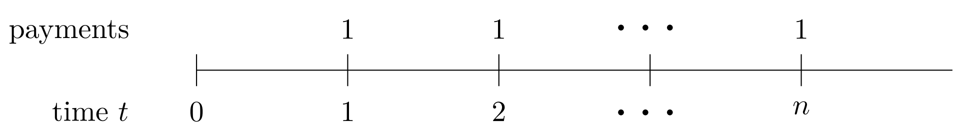
If $|r| < 1$, n can be *infinite*, because then the infinite geometric series converges (since $r^\infty = 0$):

$$1 + r + r^2 + \cdots = \frac{1}{1-r} \quad \text{for } |r| < 1 \tag{2.2}$$

Section 2.2 Annuity-Immediate Calculations

The present value of an annuity-immediate with n annual payments of 1, calculated at an annual effective interest rate i, is denoted by $a_{\overline{n}|i}$, or we can simply write $a_{\overline{n}|}$ if the value of the interest rate is clear and does not need to be specified. When referring to $a_{\overline{n}|}$, we say "**a-angle-n**." The basic formula for $a_{\overline{n}|i}$ is so important that we will derive it here:

The present value of an n-year unit annuity-immediate is the sum of the individual present values of the n payments of 1:



As of $t = 0$, the value of the first payment is v.

As of $t = 0$, the value of the second payment is v^2.

• • •

As of $t = 0$, the value of the n^{th} payment is v^n.

$$
\begin{aligned}
\text{Total present value } &= a_{\overline{n}|} = v + v^2 + \cdots + v^n \\
&= v\left(1 + v + \cdots + v^{n-1}\right) \\
&= v\frac{(1-v^n)}{1-v} = v\frac{(1-v^n)}{d} \\
&= v\frac{(1-v^n)}{iv} = \frac{1-v^n}{i}
\end{aligned}
$$

Thus we obtain the important formula:

(2.3)
$$
a_{\overline{n}|i} = \frac{1-v^n}{i}
$$

Example (2.4)

If $i = 0.05$ and $n = 10$, then $a_{\overline{10}|5\%} = \frac{1-\left(\frac{1}{1.05}\right)^{10}}{0.05} = 7.7217$.

Calculator Note

Your calculator's **TVM worksheet** can be used to calculate the value of the annuity in Example (**2.4**). The [PMT] key is used for the periodic payment of 1. The following entries give the result PV = − 7.7217:

10	[N]
5	[I/Y]
1	[PMT]
0	[FV]
[CPT]	[PV]

Note the sign convention. Positive amounts represent money paid to you, and negative amounts represent cash that you must pay out. If the applicable interest rate is 5%, you would need to pay 7.7217 now (i.e., you would have a cash flow of −7.7217) to receive 10 subsequent payments of +1.

Note: If your calculator displays an answer of −8.1078, it is in "BGN" mode. See the Calculator Note on page 69 to learn how to correct this problem.

On exams most students use the calculator's TVM functions instead of formulas whenever possible, because it saves time. You must still know the formulas, since formula knowledge is required to solve the problems, and some questions are designed so that the calculator cannot be used directly.

Exercise (2.5)

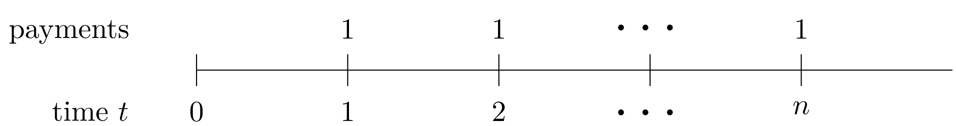
Find the value of $a_{\overline{20}|0.05}$ using the annuity formula, and then check it using your calculator's TVM functions.

Answer: 12.4622

The future value of a unit annuity-immediate with n payments is denoted by $s_{\overline{n}|}$, which is pronounced "**s-angle-n**." It is the sum of the future values (as of time n) of the n individual payments of 1.

$$s_{\overline{n}|} = (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1$$

Note that since an annuity immediate has end-of-year payments, the first payment (made at time 1) earns interest for $n-1$ **periods** (from time 1 to time n), and the last payment of 1 (made at time n) earns no interest. It is important to understand that these are the same n payments that had a present value of $a_{\overline{n}|}$ at time 0. What is different is the *valuation date*. The present value of these payments ($a_{\overline{n}|}$) has a valuation date of time 0. The future value of these same payments ($s_{\overline{n}|}$) has a valuation date of time n.

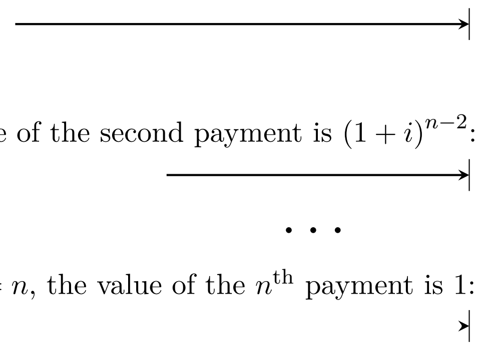


As of $t = n$, the value of the first payment is $(1+i)^{n-1}$:

As of $t = n$, the value of the second payment is $(1+i)^{n-2}$:

• • •

As of $t = n$, the value of the n^{th} payment is 1:



We could find the sum of the geometric series to develop a formula for $s_{\overline{n}|}$, but we can also find the value of $s_{\overline{n}|}$ quickly based on the formula for $a_{\overline{n}|}$. Since $a_{\overline{n}|}$ is the value of this series of n payments as of time 0, and $s_{\overline{n}|}$ is the value of the same payments n periods later (at time n), we multiply $a_{\overline{n}|}$ by $(1+i)^n$ to find $s_{\overline{n}|}$:

(2.6)
$$s_{\overline{n}|i} = (1+i)^n \cdot a_{\overline{n}|i} = (1+i)^n \cdot \frac{1-v^n}{i} = \frac{(1+i)^n - 1}{i}$$

You can use this approach to avoid excessive memorization. If you know the formula for $a_{\overline{n}|}$, you can easily write the formula for $s_{\overline{n}|}$.

Example (2.7)

If $i = 5\%$ and $n = 10$,

$$s_{\overline{10|}5\%} = 1.05^{10} \cdot a_{\overline{10|}5\%} = 1.05^{10} \times 7.7217 = 12.5779, \text{ or:}$$

$$s_{\overline{10|}5\%} = \frac{1.05^{10}-1}{0.05} = 12.5779$$

This can also be done on the financial calculator. Set N = 10, I/Y = 5, PMT = 1, and CPT FV = −12.5779. Naturally, PMT and FV have opposite signs.

Exercise (2.8)

Based on an annual effective interest rate of 6%, find $a_{\overline{15|}}$ and $s_{\overline{15|}}$.

Answers: $a_{\overline{15|}} = 9.712$ $s_{\overline{15|}} = 23.276$

To get another very useful relationship, divide both sides of Formula (**2.6**) by $(1+i)^n$:

(2.9) $$a_{\overline{n|}} = v^n \cdot s_{\overline{n|}}$$

The relationships between $a_{\overline{n|}}$ and $s_{\overline{n|}}$ in (**2.6**) and (**2.9**) are intuitive. They represent the value of n payments at time 0 and at time n, respectively, so their values differ by a factor of $(1+i)^n$, or by $(1+i)^{-n} = v^n$.

Annuities with Level Payments Other Than 1

Note that the present value or future value of *any* annuity-immediate with level payments can be found using $a_{\overline{n|}}$ and $s_{\overline{n|}}$. If an annuity-immediate has payments of amount P, its present value and future value are given by:

$$PV = P \cdot a_{\overline{n|}}$$
$$FV = P \cdot s_{\overline{n|}}$$

Example (2.10)

Find the present value of an annuity-immediate with 10 annual payments of 100, based on an annual effective interest rate of 5%:

$$100 \cdot a_{\overline{10|}} = 100 \cdot \frac{1-\left(\frac{1}{1.05}\right)^{10}}{0.05} = 100 \times 7.7217 = 772.17$$

Exercise (2.11)

Find the present value of an annuity-immediate with 30 annual payments of 500, based on an annual effective interest rate of 8%.

Answer: 5,628.89

Section 2.3 Perpetuities

A **perpetuity** is an annuity with payments that continue forever. The present value of a perpetuity-immediate that pays 1 per period is denoted by $a_{\overline{\infty}|}$.

(2.12) $$a_{\overline{\infty}|} = v + v^2 + v^3 + \cdots$$

If we write $a_{\overline{\infty}|}$ as a limit, we obtain the following formula:

(2.13) $$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|} = \lim_{n\to\infty} \frac{1-v^n}{i} = \frac{1}{i}$$

Example (2.14)

If $i = 5\%$, then $a_{\overline{\infty}|} = \frac{1}{0.05} = 20$.

Exercise (2.15)

Find the present value of a unit perpetuity-immediate with $i = 8\%$.

Answer: 12.50

You can think of a perpetuity as a "savings account" that pays out the interest earned each year but never pays out any of the principal. Using that concept, we can develop Formula (**2.13**) using simple algebra. Let X be the value of a perpetuity-immediate. At $t = 0$, the "savings account" has a balance of X. At $t = 1$, the balance is $X \cdot (1 + i) = X + X \cdot i$, and the interest $(X \cdot i)$ is paid out, leaving a balance of X in the account. Each year, immediately after the interest payment of $(X \cdot i)$, the balance will again be X. (This is logical, because there are always an infinite number of future payments, and X was defined as the value of the perpetuity, i.e., the value of an infinite number of future payments.) If the perpetuity's annual payment amount is 1, then we have:

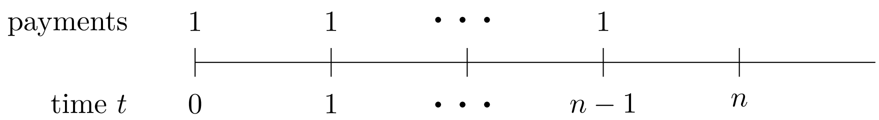
$$X \cdot i = 1 \longrightarrow X = 1/i$$

Thus X, the value of our unit perpetuity-immediate, has the same value we found for $a_{\overline{\infty}|}$ in Formula (**2.13**).

Note: We cannot write a formula for the future value of a perpetuity (which would be written $s_{\overline{\infty}|}$). That would be the perpetuity's value as of the date of its last payment, and there is no "last payment" under a perpetuity.

Section 2.4 Annuity-Due Calculations

The present value of an n-period unit **annuity-due** is denoted by $\ddot{a}_{\overline{n}|}$, which is pronounced "**a-double-dot-angle-n**."



As of $t = 0$, the value of the first payment is 1.

As of $t = 0$, the value of the second payment is v.

• • •

As of $t = 0$, the value of the n^{th} payment (which occurs at $t = n - 1$) is v^{n-1}.

Summing the present values of these payments, we have:

$$\ddot{a}_{\overline{n}|} = 1 + v + \cdots + v^{n-1} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{d}$$

Thus:

(2.16)
$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d}$$

Another way to develop this formula is to recognize that each of the n payments in this annuity due occurs *one period earlier* than the corresponding payment under an n year annuity immediate. As a result, $\ddot{a}_{\overline{n}|}$ has a value that is larger than $a_{\overline{n}|}$ by a factor of $(1 + i)$:

(2.17)
$$\ddot{a}_{\overline{n}|} = (1 + i) \cdot a_{\overline{n}|} = (1 + i) \cdot \frac{1 - v^n}{i} = \frac{1 - v^n}{i/(1 + i)} = \frac{1 - v^n}{d}$$

The formula for $\ddot{a}_{\overline{n}|}$ is easy to remember, since it is obtained by taking the equation for $a_{\overline{n}|}$ and replacing the i in the denominator by d. As a memory aid, you might use the fact that the words "immediate" and "due" begin with the letters "i" and "d," and that i and d are their respective denominators. This pattern of denominators also applies to formulas for the *future* value of an annuity-due and the present value of a perpetuity-due:

(2.18)
$$\ddot{s}_{\overline{n}|} = \frac{(1 + i)^n - 1}{d}$$

(2.19)
$$\ddot{a}_{\overline{\infty}|} = \frac{1}{d}$$

We can develop some useful relationships between annuities-immediate and annuities-due by general reasoning. For example, an n-period annuity-due consists of the same payments as an $(n-1)$-period annuity-immediate plus a payment of 1 at time 0, so we can write:

(2.20)
$$\ddot{a}_{\overline{n}|} = a_{\overline{n-1}|} + 1$$

For a perpetuity, this becomes:

(2.21)
$$\ddot{a}_{\overline{\infty}|} = a_{\overline{\infty}|} + 1$$

The future value of an n-period annuity-immediate is equal to the future value of an $(n-1)$-period annuity-due plus an n^{th} payment at time n (the valuation date):

(2.22)
$$s_{\overline{n}|} = \ddot{s}_{\overline{n-1}|} + 1$$

Example (2.23)

Given $i = 5\%$ and $n = 10$, find $\ddot{a}_{\overline{n}|}$ directly, and check your answer using the relationship between $\ddot{a}_{\overline{n}|}$ and $a_{\overline{n}|}$.

Solution.

$$\begin{aligned}\ddot{a}_{\overline{10}|} &= \frac{1-v^n}{d} = \frac{1-\left(\frac{1}{1.05}\right)^{10}}{\left(\frac{0.05}{1.05}\right)} \\ &= 8.1078\end{aligned}$$

Check:
From Example (**2.4**), $a_{\overline{10}|} = 7.7217$:

$$\begin{aligned}\ddot{a}_{\overline{10}|} &= (1+i)\cdot a_{\overline{10}|} = 1.05(7.7217) \\ &= 8.1078\end{aligned}$$

Formula (**2.20**) provides another way to find $\ddot{a}_{\overline{10}|}$:

$$\begin{aligned}\ddot{a}_{\overline{10}|} &= a_{\overline{9}|} + 1 = \frac{1-1.05^{-9}}{0.05} + 1 = 7.1078 + 1 \\ &= 8.1078\end{aligned}$$

Exercise (2.24)

Calculate $\ddot{a}_{\overline{15}|}$ and $\ddot{s}_{\overline{15}|}$ based on an annual effective interest rate of 6%.

Answers: $\ddot{a}_{\overline{15}|6\%} = 10.295$ $\ddot{s}_{\overline{15}|6\%} = 24.673$

Calculator Note

Annuity-due calculations are done with the calculator set to the BGN (begin) mode to reflect that payments are made at the *beginning* of the period. The letters BGN appear above the [PMT] key. If you key in [2nd] [BGN] you will see either BGN or END in the calculator's display. You can then change to the other mode by pressing [2nd] and [SET] (the 2$^{\text{nd}}$ function of the [ENTER] key). *Remember that you can leave the* BGN/END *menu by pressing the* [CE/C] *key or by pressing* [2nd] [QUIT].

You can tell whether your calculator is set to BGN or END mode by looking at the upper right of the screen. If "BGN" appears in small letters at the upper right, the calculator is in BGN (annuity-due) mode. If BGN does not appear on the screen, it is in END (annuity-immediate) mode.

It is very important on actuarial exams to be aware of your calculator's BGN/END *mode. The majority of problems require* END *mode. If you do a* BGN *mode problem and do not reset the calculator to* END *mode, you will have trouble on subsequent problems. Many students avoid this difficulty by keeping their calculators set to* END *at all times. When they need to find the value of an annuity-due, they calculate the value of the corresponding annuity-immediate, and then multiply by* $(1+i)$.

Example (2.25)

Calculate $\ddot{a}_{\overline{10}|5\%}$ using the BA II Plus's TVM worksheet.

Solution.
Set the calculator to BGN mode. Set N = 10, I/Y = 5, PMT = −1, and FV = 0.
CPT PV = 8.1078.

We could have entered the same values with the calculator in END mode. In that case, we would CPT PV = 7.7217, the value of an annuity-immediate. Then multiply by 1.05 to get the answer: $\ddot{a}_{\overline{10}|} = 8.1078$.

Exercise (2.26)

Calculate $\ddot{a}_{\overline{15}|6\%}$ using the calculator's TVM worksheet.

Answer: 10.295

Reminder: Be sure to return your calculator to END mode.

Section 2.5 Continuously Payable Annuities

A continuous unit annuity pays a total of 1 per year, but spreads the payment evenly throughout the year by making a continuous stream of payments at a constant rate of 1 per year. We can think of this **continuously payable annuity** as making a payment of $(1 \cdot dt)$ in each infinitesimal time interval of length dt. The present value (at time 0) of a payment of $(1 \cdot dt)$ made at time t is $\left(v^t \cdot dt\right)$.

An n-year continuous unit annuity, which makes continuous payments at a rate of 1 per year from time 0 to time n, is denoted by $\bar{a}_{\overline{n}|}$ ("**a-bar-angle-n**"). Its present value is found by integrating the present value of its rate of payment $\left(v^t \cdot dt\right)$ from time 0 to time n:

$$\bar{a}_{\overline{n}|} = \int_{t=0}^{n} v^t \cdot dt = \left.\frac{v^t}{\ln(v)}\right|_{t=0}^{n}$$

$$= \frac{v^n - v^0}{\ln\left(\frac{1}{1+i}\right)} = \frac{v^n - 1}{-\delta} = \frac{1 - v^n}{\delta}$$

Note: In this derivation, we used the relation

$$\ln(v) = \ln\left(\frac{1}{1+i}\right) = -\ln(1+i) = -\delta.$$

The final result is:

(2.27)
$$\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta} = \frac{i}{\delta} \cdot a_{\overline{n}|}$$

Note that this formula follows the pattern we observed in the formula for $\ddot{a}_{\overline{n}|}$. We can find $\bar{a}_{\overline{n}|}$ by changing the denominator of $a_{\overline{n}|}$ from i to δ (just as we changed it from i to d for $\ddot{a}_{\overline{n}|}$). This is equivalent to multiplying $a_{\overline{n}|}$ by $\frac{i}{\delta}$.

Similarly, for the *future* value of a continuously payable unit annuity, we have:

(2.28)
$$\bar{s}_{\overline{n}|} = (1+i)^n \cdot \bar{a}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta} = \frac{i}{\delta} \cdot s_{\overline{n}|}$$

For a continuously payable unit *perpetuity*:

(2.29)
$$\bar{a}_{\overline{\infty}|} = \frac{1}{\delta} = \frac{i}{\delta} \cdot a_{\overline{\infty}|}$$

Example (2.30)

For $i = 5\%$, calculate $\bar{a}_{\overline{10}|}$ and $\bar{s}_{\overline{10}|}$.

Solution.

$$\bar{a}_{\overline{10}|} = \frac{1 - v^{10}}{\delta} = \frac{1 - \left(\frac{1}{1.05}\right)^{10}}{\ln(1.05)}$$
$$= 7.9132$$

This can be checked by using the $\frac{i}{\delta}$ adjustment factor and the fact that $a_{\overline{10}|} = 7.7217$:

$$\bar{a}_{\overline{10}|} = \frac{i}{\delta} \cdot a_{\overline{10}|} = \frac{0.05}{\ln(1.05)}(7.7217)$$
$$= 7.9132$$

For $\bar{s}_{\overline{10}|}$, we have:

$$\bar{s}_{\overline{10}|} = \frac{1.05^{10} - 1}{\ln(1.05)} = 12.8898$$

We can check this value by accumulating $\bar{a}_{\overline{10}|}$ for 10 years (i.e., moving the valuation date from $t = 0$ to $t = 10$):

$$\bar{s}_{\overline{10}|} = 1.05^{10} \cdot \bar{a}_{\overline{10}|} = 1.6289 \times 7.9132$$
$$= 12.8898$$

Calculator Note

The BA II Plus has 10 memories (in addition to the memories associated with the TVM functions and the various worksheets). These 10 memories are numbered 0 to 9. To store the currently displayed value in (for example) Memory 2, press STO 2. Then to recall that value for use in a calculation, press RCL 2.

When solving a problem, you will frequently calculate an intermediate result that will be needed later in solving that problem. It is best to use your calculator's memories to store such values. It may also be useful to write down a 3- or 4-digit *approximation* of the intermediate result as a record of your work. But re-entering that *approximate* value into the calculator in place of the *calculated* value wastes time and loses accuracy. Instead, *use the calculator's memories* to maintain the *full precision* of these values.

Exercise (2.31)

For $i = 6\%$, calculate $\bar{a}_{\overline{15}|}$ and $\bar{s}_{\overline{15}|}$.

Answers: $\bar{a}_{\overline{15}|} = 10.0008$ $\bar{s}_{\overline{15}|} = 23.9675$

Example (2.32)

A 20-year continuous stream of payments consists of payments at a rate of 3,000 per year for the first 10 years, then at a rate of 2,000 per year from $t = 10$ to $t = 20$. At an interest rate of 6% convertible monthly, what is the present value of this payment stream?

Solution.

The payment stream can be broken into two parts: continuous payments at a rate of 2,000 per year for the full 20-year period, and continuous payments of 1,000 per year for just the first 10 years. The total present value is:

$$2,000 \cdot \bar{a}_{\overline{20}|} + 1,000 \cdot \bar{a}_{\overline{10}|} = 2,000 \cdot \frac{1 - v^{20}}{\delta} + 1,000 \cdot \frac{1 - v^{10}}{\delta}$$

In order to evaluate this expression, we need values for v and δ:

$$v = \frac{1}{1+i} = \frac{1}{\left(1 + \frac{i^{(12)}}{12}\right)^{12}} = \frac{1}{\left(1 + \frac{0.06}{12}\right)^{12}} = \frac{1}{1.06168}$$
$$= 0.9419$$

$$\delta = \ln(1+i) = \ln(1.06168)$$
$$= 0.05985$$

Using these values, the present value of this payment stream is:

$$2,000 \cdot \frac{1 - v^{20}}{\delta} + 1,000 \cdot \frac{1 - v^{10}}{\delta} = 2,000 \cdot \frac{1 - 0.9419^{20}}{0.05985} + 1,000 \cdot \frac{1 - 0.9419^{10}}{0.05985}$$
$$= 2,000 \times 11.6607 + 1,000 \times 7.5249$$
$$= 30,846.44$$

Exercise (2.33)

An account pays interest at a continuously compounded rate of 0.05 per year. Continuous deposits are made to the account at a rate of 1,000 per year for 6 years, and then at a rate of 2,000 per year for the next 4 years. What is the account balance at the end of 10 years?

Answer: 17,402.48

Section 2.6 Basic Annuity Problems for Calculator Practice

We can solve for each of the variables PMT, PV, FV, I/Y, and N using the BA II Plus. In this section we give an example of solving for each of these using the BA II Plus calculator. In each case, a formula is also given, indicating how the problem can be solved using annuity functions.

Example (2.34)

Solving for PMT
A loan for 20,000 is to be repaid by 5 year-end payments with interest at an annual effective rate of 12%. What is the amount of the annual payment?

Solution.
The 5 payments must have a present value equal to the amount of the loan:

$$P \cdot a_{\overline{5}|12\%} = 20,000$$

Set N = 5, I/Y = 12, and PV = 20,000. CPT PMT = −5,548.19.

The annual payment is 5,548.19.

Exercise (2.35)

A loan for 15,000 is to be repaid by 8 year-end payments with interest at an annual effective rate of 10%. What is the amount of the annual payment?

Answer: 2,811.66

Example (2.36)

Solving for PMT
You have a 5,000 balance in an account earning a 4.5% annual effective rate. You want to increase your balance to 20,000 at the end of 12 years by making a level deposit of D at the beginning of each of the next 12 years. Find D, the required level annual deposit.

Solution.

$$5,000 \times 1.045^{12} + D \cdot \ddot{s}_{\overline{12}|4.5\%} = 20,000$$

Put the calculator in BGN mode. Set N = 12, I/Y = 4.5, PV = −5,000, and FV = 20,000. CPT PMT = −712.91.

The level annual deposit is 712.91.

The problem of Example (**2.36**) could also have been solved with the calculator in END mode. In that case, you would enter the same values, but would need to adjust the result:

Set N = 12, I/Y = 4.5, PV = −5,000, and FV = 20,000. CPT PMT = −744.99.

744.99 is the amount you would need to deposit at the *end* of each year. Since this problem involves deposits made one year earlier (at the *beginning* of each year), the deposits should be smaller by a factor of $1/(1+i)$: $\frac{744.99}{1.045} = 712.91$.

Exercise (2.37)

What is the required level deposit in (**2.36**) if the current balance is 4,000 and the annual effective interest rate is 6%?

Answer: 668.33

If you have changed your calculator setting to BGN mode, be sure to reset it to END mode.

Example (2.38)

Solving for PV

You wish to make a deposit now to an account earning a 5% annual effective rate so that you can withdraw 1,000 at the end of each of the next 15 years. How much should you deposit today?

Solution.

$$D = 1,000 \cdot a_{\overline{15}|5\%}$$

Set N = 15, I/Y = 5, and PMT = 1,000. CPT PV = −10,379.66.

You should deposit 10,379.66.

Exercise (2.39)

What would be the required deposit in Example (**2.38**) if you wanted 20 years of withdrawals instead of 15?

Answer: 12,462.21

Example (2.40)

Solving for FV

An account earning a 5% annual effective rate has a current balance of 6,000. If a deposit of 1,500 is made at the end of each year for 20 years, what will be the balance in the account at the end of 20 years?

Solution.

$$Bal_{20} = 6,000 \cdot 1.05^{20} + 1,500 \cdot s_{\overline{20}|5\%}$$

Set N = 20, I/Y = 5, PV = −6,000, and PMT = −1,500. CPT FV = 65,518.72.

The balance at the end of 20 years will be 65,518.72.

Exercise (2.41)

What would be the ending balance in Example (**2.38**) if the interest rate were 6% instead of 5%?

Answer: 74,421.20

Example (2.42)

Solving for I/Y

You have borrowed 15,000 and agreed to repay the loan with 5 level annual payments of 4,000, with the first payment occurring one year from the date of the loan. What annual effective interest rate are you paying?

Solution.

$$15,000 = 4,000 \cdot a_{\overline{5}|i}$$

Set N = 5, PV = 15,000, and PMT = −4,000. CPT I/Y = 10.42.

You are paying interest at a 10.42% annual effective rate.

Note that PV is positive, since it represents cash you received; PMT *is negative, because it is cash that you must pay. If you forget the minus sign, the* BA II Plus *will give an error message when you press* CPT I/Y.

Exercise (2.43)

What would the interest rate be in Example (**2.42**) if the annual loan payment were 4,300?

Answer: 13.34%

Note: The equations of value in Example (**2.42**) *and Exercise* (**2.43**) *involve* 5^{th} *degree polynomials in* v *or* $(1+i)$. *There is no formula to solve for* i *directly. Instead, the TVM worksheet "iterates" (uses successive approximations) until it finds a value for* i *that satisfies the equation within a small margin of error.*

Example (2.44)

Solving for N

You want to accumulate at least 20,000 in an account earning a 5% annual effective rate. You will make a level deposit of 1,000 at the beginning of each year for n years. What is the value of n? What is the account balance after n years?

Solution.

The equation of value is $20,000 = 1,000 \cdot \ddot{s}_{\overline{n}|5\%}$.

We can find n using algebra, as follows:

$$20,000 = 1,000 \cdot \ddot{s}_{\overline{n}|5\%} = 1,000 \cdot \frac{1.05^n - 1}{0.05/1.05}$$

$$20 \cdot (0.05/1.05) + 1 = 1.05^n$$

$$n = \frac{\ln[20 \cdot (0.05/1.05) + 1]}{\ln 1.05} = 13.71$$

The *computed* value for n is 13.71. This answer means that 13 payments are not enough, and a 14^{th} payment is required to reach 20,000. Thus $n = 14$.

$$1,000 \cdot \ddot{s}_{\overline{14}|5\%} = 1,000 \cdot \frac{1.05^{14} - 1}{0.05/1.05} = 20,578.56$$

The account balance after 14 years is 20,578.56.
To solve by calculator, put the BA II Plus into BGN mode and set I/Y = 5, PMT = −1,000, and FV = 20,000. CPT N = 13.71.

Now enter 14 for N and CPT FV = 20,578.56.

(Or, to solve the problem in END mode, make the same entries, except enter −1,050 for PMT ($1,050 = 1,000 \times 1.05$) to reflect that a payment of 1,050 at year-end is equivalent to a payment of 1,000 at the beginning of the year.)

A note about the answer to Example (2.44):

It is important to understand that the annuity formula $a_{\overline{n}|} = \frac{1-v^n}{i}$ is valid only if n is an integer, because it is based on the geometric series formula, which requires n to be an integer. The calculated value of 13.71 in Example (**2.44**) does not mean that the account balance is 20,000 at time 13.71 years. Rather, it simply tells us that the balance will be less than 20,000 at time 13 (after 13 payments), and it will be more than 20,000 at time 14 (after 14 payments).

We can calculate the amount of the *partial* payment needed at the beginning of the 14^{th} year to produce a balance of *exactly* 20,000 at the end of 14 years. We have already calculated that the balance at time 14 will be 20,578.56. This suggests that we could have deposited $\frac{578.56}{1.05} = 551.01$ less at the beginning of the 14^{th} year (at time 13). Thus a final payment of 448.99 ($= 1,000 - 551.01$) at time 13 would produce a balance of exactly 20,000 at time 14.

Alternatively, if the deposit at the beginning of the 14^{th} year is the full 1,000, we can calculate when during the 14^{th} year the balance will reach 20,000. After the deposit at time 13, the balance is $1,000 \cdot \ddot{s}_{\overline{13}|5\%} + 1,000 = 19,598.63$. Let p be the fraction of a year required for this amount to grow to 20,000. Then:

$$19,598.63 \cdot 1.05^p = 20,000$$

$$1.05^p = 20,000/19,598.63 = 1.02048$$

$$p = \frac{\ln 1.02048}{\ln 1.05} = 0.4155$$

The balance will reach 20,000 after 13.4155 years.

Or, using the **BA II Plus**, set I/Y = 5, PV = −19,598.63, PMT = 0, and FV = 20,000. CPT N = 0.4155.

In this case, the non-integer value of N is valid. Because there are no periodic payments (PMT = 0), there is no geometric series, so N does not have to be an integer.

Exercise (2.45)

How many payments (n) would be needed in Example (**2.44**) if the interest rate were 6%? What payment amount at the beginning of the n^{th} year would produce a balance of exactly 20,000 at time n?

Answers: 13 985.79

Section 2.7 Annuities with Varying Payments

Not all series of payments are level. In practice, it is quite common to encounter **annuities with varying payments**, such as the following:

Series of Payments:	***Payments Made:***	***Type of Annuity Sequence:***
500, 0, 200, 300	At end of period	Irregular
1, 2, 3, 4	At end of period	Arithmetic increasing
4, 3, 2, 1	At end of period	Arithmetic decreasing
$1, 1.05, 1.1025 = 1.05^2$	Beginning of period	Geometric

In interest theory, there are formulas for the last three sequences presented here, and they will be covered in the following sections. But if there are only four or five terms to input, you can calculate the annuity's value more quickly by using your calculator's **Cash Flow worksheet**.

For example, if $i = 0.05$, you can find the present value of the increasing annuity-immediate $\{1, 2, 3, 4\}$ by using the [CF] (Cash Flow) and [NPV] (Net Present Value) keys:

First press the [CF] key to activate the Cash Flow worksheet. Then press [2nd] [CLR WORK] to clear any cash flow values that were previously entered. You will see a prompt for the value of CF_0, the cash flow at time 0. In this case, there is no payment until time 1, so leave the CF_0 value at 0 and press the [↓] key. You will see "C01," requesting the cash flow at time 1. Press 1 and [ENTER]. Scroll down again, and there will be a new prompt: "F01." This is a request for the number of times (frequency) this value is repeated. The default value is 1, and if you scroll down again, the value of 1 will be assumed with no entry. After you scroll down, you will be prompted for the value of C02. Press 2 and [ENTER]. Repeat this process until all four cash flow values have been entered. Then calculate the NPV at 5% with the following keystrokes:

[NPV] [5] [ENTER] [↓] [CPT]

The display will show the answer: NPV = 8.6488

To find the present value of the first series $(500, 0, 200, 300)$ at $i = 0.05$, enter the cash flows and use the NPV function with I = 5. Result: NPV = 895.77.

Note $CF_0 = 0$, because CF_0 is the payment at $t = 0$. The first payment is at the end of the first period ($t = 1$). Thus, $CF_0 = 0$, C01=500, C02=0, C03=200, and C04=300.

To calculate the present value of the last series in the table, enter 1 for CF_0. (Payments are made at the beginning of each period, so the first payment is at $t = 0$.) Then enter 1.05 for C01, and 1.1025 for C02. At a 6% annual effective interest rate, the present value is 2.9718.

Section 2.8 Increasing Annuities with Terms in Arithmetic Progression

An annuity whose n payments are 1, 2, 3, ..., n is called a **unit increasing annuity**. If payments are made at the *end* of each period, it is an increasing annuity-immediate. The present value of this annuity is denoted by $(Ia)_{\overline{n}|}$ and, of course, $(Ia)_{\overline{n}|}$ is equal to the present value of the annuity's payments:

$$(Ia)_{\overline{n}|} = v + 2v^2 + 3v^3 + \ldots + n \cdot v^n.$$

To develop a practical formula for this function, first note that the payment pattern is as follows:

payments		1	2	3	• • •	n
time t	0	1	2	3	• • •	n

These same payments can be arranged as shown in the following table. This arrangement allows us to write a formula for the present value of the payments in each line. We can then sum those present values to create a formula for the value of $(Ia)_{\overline{n}|}$:

time t	1	2	3	$\cdots$	n	Present Value at $t = 0$
payments	1	1	1	$\cdots$	1	$\frac{1-v^n}{i}$
		1	1	$\cdots$	1	$v \cdot \frac{1-v^{n-1}}{i} = \frac{v-v^n}{i}$
			1	$\cdots$	1	$v^2 \cdot \frac{1-v^{n-2}}{i} = \frac{v^2-v^n}{i}$
					$\cdots$	$\cdots$
					1	$v^{n-1} \cdot \frac{1-v}{i} = \frac{v^{n-1}-v^n}{i}$
Total	1	2	3		n	$\frac{\sum_{t=0}^{n-1} v^t - n \cdot v^n}{i} = \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{i}$

The result is:

(2.46)
$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{i}$$

Example (2.47)

Let $i = 5\%$ and $n = 4$. Then the increasing annuity payments are 1, 2, 3, 4, and the present value is:

$$(Ia)_{\overline{4}|} = \frac{\ddot{a}_{\overline{4}|} - 4 \cdot v^4}{0.05} = 8.6488$$

Note that 8.6488 is the same value we calculated for this sequence of payments in the previous section using the Cash Flow worksheet.

Exercise (2.48)

Find $(Ia)_{\overline{15}|i=6\%}$.

Answer: 67.2668

As with level annuities, to create a formula for an increasing unit annuity-*due* or a continuously payable increasing annuity, we just change the denominator:

(2.49) $$(I\ddot{a})_{\overline{n}|} = \frac{i}{d} \cdot (Ia)_{\overline{n}|} = (1+i) \cdot (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{d}$$

(2.50) $$(I\bar{a})_{\overline{n}|} = \frac{i}{\delta} \cdot (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{\delta}$$

In the case of an increasing perpetuity, the present value is the limit of the appropriate formula as n approaches infinity:

(2.51) $$(Ia)_{\overline{\infty}|} = \lim_{n\to\infty} \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{i} = \frac{\frac{1}{d} - 0}{i} = \frac{1}{id}$$

(2.52) $$(I\ddot{a})_{\overline{\infty}|} = \lim_{n\to\infty} \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{d} = \frac{\frac{1}{d} - 0}{d} = \frac{1}{d^2}$$

(2.53) $$(I\bar{a})_{\overline{\infty}|} = \lim_{n\to\infty} \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{\delta} = \frac{\frac{1}{d} - 0}{\delta} = \frac{1}{\delta d}$$

Note: In Formulas (**2.50**) *and* (**2.53**), *payments are made continuously, but the increases occur annually. That is, continuous payments are made at a rate of 1 per year during the* 1^{st} *year, 2 per year during the* 2^{nd} *year, etc.*

Example (2.54)

Let $i = 5\%$ and $n = 4$.

Then, $(I\ddot{a})_{\overline{4}|5\%} = 1.05 \cdot (Ia)_{\overline{4}|5\%} = 1.05 \times 8.6488 = 9.0812$.

Note: 8.6488 is the value calculated for $(Ia)_{\overline{4}|5\%}$ in Example (**2.47**).

Exercise (2.55)

Find $(I\ddot{a})_{\overline{15}|6\%}$.

Answer: 71.3028

The *future* value of an increasing unit annuity-immediate is denoted by $(Is)_{\overline{n}|}$. One can avoid excessive memorization of formulas by using the relationship $(Is)_{\overline{n}|} = (1+i)^n \cdot (Ia)_{\overline{n}|}$. (The value as of the date of the n^{th} payment equals the value as of time 0, $(Ia)_{\overline{n}|}$, accumulated n periods to time n.) Below, we show the commonly used expressions for calculating $(Is)_{\overline{n}|}$, $(I\ddot{s})_{\overline{n}|}$, and $(I\bar{s})_{\overline{n}|}$.

(2.56)
$$(Is)_{\overline{n}|} = (1+i)^n \cdot (Ia)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

(2.57)
$$(I\ddot{s})_{\overline{n}|} = (1+i)^n \cdot (I\ddot{a})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} = \frac{i}{d} \cdot (Is)_{\overline{n}|}$$

(2.58)
$$(I\bar{s})_{\overline{n}|} = (1+i)^n \cdot (I\bar{a})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{\delta} = \frac{i}{\delta} \cdot (Is)_{\overline{n}|}$$

!

The number of formulas here appears overwhelming, but the situation is relatively simple, and the adjustment factors are the same as for level annuities.

If you can calculate $(Ia)_{\overline{n}|}$, then to obtain the value of any other type of increasing annuity, simply do a multiplication:

Multiply by: $(1+i)^n$ to convert $(Ia)_{\overline{n}|}$ to $(Is)_{\overline{n}|}$.

Multiply by i/d to convert to an annuity-due.

Multiply by i/δ to convert to a continuously-payable annuity.

Extensive memorization is not required!

Midterm 1

Questions

1. An amount of 1,000 is invested today at an interest rate of 5% per annum. How much more interest will it earn during the 4^{th} year if it earns 5% compound interest than if it earns 5% simple interest?

(A) 0 (B) 4.33 (C) 7.88 (D) 10.88 (E) 12.22

2. Theresa deposited 1,000 into a fund at the end of each year for 20 years. When she received a bonus from her employer at the end of the 5^{th}year, she deposited an additional amount A into the fund. The fund credits interest at a 5% annual effective rate.

At the end of 20 years, Theresa withdrew an Amount A (equal to the extra amount she deposited after 5 years). She was then able to make annual withdrawals of 3,000 at the end of each year for 20 years (first withdrawal of 3,000 at time 21).

To the nearest 100, what is the value of A?

(A) 3,000 (B) 3,600 (C) 4,000 (D) 4,500 (E) 4,900

3. An annual-payment annuity-due pays an initial amount of 2 per year, with the payments increasing by 5.25% every four years. The annuity consists of 40 annual payments.

Using an annual effective interest rate of 3%, calculate the value of this annuity as of the date of the last payment.

(A) 58 (B) 76 (C) 125 (D) 183 (E) 197

4. Andy Skorz deposits X into an account, and then deposits $2X$ into the account 2 years later. The account earns a 10% annual effective interest rate for the first 3 years, and a nominal interest rate of 5% convertible semi-annually thereafter. At the end of 6 years, the account balance is 200,000.

How much interest did Andy earn during the 2^{nd} year?

(A) 5,373 (B) 5,613 (C) 5,678 (D) 5,786 (E) 5,957

5. Given $d^{(4)} = 0.05$ and $A(1.5) = 100$, find $A(0)$, the value at time 0.

(A) 92.7 (B) 93.0 (C) 93.1 (D) 93.8 (E) 94.0

6. A perpetuity-immediate has 32 initial quarterly payments of 20, followed by quarterly payments of 25 in the 9^{th} and later years. Find this perpetuity's present value at a nominal interest rate of 16% convertible quarterly.

(A) 982 (B) 877 (C) 715 (D) 610 (E) 536

7. If $i^{(4)} = 0.05$, find $d^{(2)}$.

(A) 4.6% (B) 4.7% (C) 4.8% (D) 4.9% (E) 5.0%

8. At time 0, Allison decided that she will purchase an item at time 10. At time 0 the item's price was 1,000. She began depositing an amount A at the beginning of each year into an account that earned interest at a 6% annual effective rate, so that she would have 1,000 at time 10.

After 5 years, Allison realized that, due to inflation, the item's price had increased by 4% each year (since time 0). Assuming that the price would continue to increase by 4% each year, she added x to each of her annual deposits, starting at time 5, to accumulate an amount equal to the projected price at time 10.

Calculate x.

(A) 22.49 (B) 23.24 (C) 23.96 (D) 24.50 (E) 25.40

9. A perpetuity-immediate pays 10 annually for the first 20 years. After 20 years, the payment amount decreases by 1 each year until it reaches an amount of 1. Annual payments of 1 continue forever. The annual effective rate of interest is 6%. Calculate the present value of this perpetuity.

(A) 99 (B) 105 (C) 129 (D) 136 (E) 140

10. Suppose you can earn interest at a 10% annual effective rate for the next 10 years and at 6% convertible semi-annually for the 10 years after that. What would be the required investment today to accumulate to 1,000,000 in 20 years?

(A) 215,285 (B) 213,466 (C) 213,270 (D) 212,115 (E) 211,980

11. Annual deposits of 500 are made at the beginning of each year for 30 years to an account earning an annual effective rate of 7%. The interest earned each year is reinvested in another fund that earns a 4.5% annual effective rate. At the end of the 30 years, what is the total accumulated value of the 30 payments and the reinvested interest?

(A) 15,000 (B) 26,250 (C) 31,500 (D) 41,250 (E) 42,575

12. At time 0 a deposit of 100 is made to an account that accumulates at a varying force of interest defined by:

$$\delta_t = \frac{2}{20-t} \qquad \text{for } 0 \le t \le 10$$

Also at time 0, a deposit of 100 is made to a second account that accumulates at a varying force of interest defined by:

$$\delta_t = \frac{1}{8+t} \qquad \text{for } t \ge 0$$

At what time between 0 and 10 will the balances in the two accounts be equal?

(A) 2.73 (B) 3.27 (C) 5.97 (D) 7.32 (E) 9.27

13. To fund her retirement, Bernice decided to save 10,000 at the end of each year for 30 years in an account that earns an annual effective interest rate of 6%. Due to a period of unemployment, Bernice did not make deposits in years 10, 11, and 12. She resumed making deposits of 10,000 in years 13 and 14, but then realized that she would need to increase the amount of her deposits in order to reach her original savings goal. As a result, Bernice then increased her deposits for years 15 through 30 by an amount A in order to reach her original savings goal at the end of 30 years.

To the nearest 100, what is the value of A?

(A) 2,900 (B) 3,500 (C) 5,300 (D) 7,300 (E) 9,900

14. Violet plans to deposit 1,000 at the end of each year for 20 years into a savings account that earns an annual effective interest rate of 6%. After 20 years, she will withdraw an amount from the account at the end of each year until the account balance is 0.

Violet determines that if she withdraws A at the end of each year, the balance will reach 0 at the end of 10 years, but if she withdraws B at the end of each year, the balance will reach 0 at the end of 12 years.

What is the value of A–B?

(A) 525 (B) 545 (C) 571 (D) 591 (E) 610

15. At $t = 0$, \$1,000 is deposited into an account that earns interest at an 8% annual effective rate. Also at $t = 0$, an amount X is deposited into another account that earns simple interest at a rate of i per year.

At $t = 2$, the balances in the two accounts are equal. At $t = 6$, the balances in the two accounts are again equal.

Calculate i.

(A) 10.1% (B) 10.4% (C) 10.7% (D) 11.0% (E) 11.3%

Solutions

1. The 4^{th} year is the period between time 3 and time 4. The interest earned during the 4^{th} year is the accumulated value at time 4 minus the accumulated value at time 3.

   Using compound interest: $1,000 \cdot 1.05^4 - 1,000 \cdot 1.05^3 = 57.88$

   Using simple interest: $1,000(1 + 4(0.05)) - 1,000(1 + 3(0.05)) = 50$

   (For simple interest, we could just note that the amount of interest earned each year is equal to 5% of the unchanging principal of 1,000.)

   The difference is $57.88 - 50 = 7.88$.

   Answer C

2. The accumulated value of the 20 deposits of 1,000 is:
$$1,000s_{\overline{20|}} = 1,000(1.05^{20} - 1)/0.05 = 33,065.95$$
   (Or set N = 20, I/Y = 5, PV = 0, and PMT = −1,000. CPT FV = 33,065.95.)

   The amount needed at time 20 to fund 20 annual withdrawals of 3,000 is:
$$3,000a_{\overline{20|}} = 3,000(1 - 1.05^{-20})/0.05 = 37,386.63$$
   (Or set N = 20, I/Y = 5, PMT = 3,000, and FV = 0. CPT PV = −37,386.63)

   The difference between the accumulated amount at 20 years (33,065.95) and the amount needed to fund the withdrawals (37,386.63) has to be funded by the deposit of A at time 5 and the withdrawal of the same amount A at time 20. The following equation sets the effect of this deposit and withdrawal equal to the additional amount that is needed at time 20:
$$A(1.05)^{15} - A = 37,386.63 - 33,065.95 = 4,320.68$$
$$A = 4,320.68/(1.05^{15} - 1) = 4,004.60$$

   Answer C

3. You could set up the present value calculation with 10 separate 4-payment annuities-due.
$$PV = 2 \cdot \ddot{a}_{\overline{4|}0.03} + 2(1.0525)\ddot{a}_{\overline{4|}0.03} \cdot v^4 + 2(1.0525)^2\ddot{a}_{\overline{4|}0.03} \cdot v^8 + \cdots + 2(1.0525)^9\ddot{a}_{\overline{4|}0.03} \cdot v^{36}$$
$$= 2 \cdot \ddot{a}_{\overline{4|}0.03}\left[1 + 1.0525 \cdot v^4 + 1.0525^2 \cdot v^8 + \cdots + 1.0525^9 \cdot v^{36}\right]$$
$$= 2 \cdot \ddot{a}_{\overline{4|}0.03}\left[\frac{1 - \left(1.0525 \cdot v^4\right)^{10}}{1 - 1.0525 \cdot v^4}\right]$$
$$= 7.6572 \times 7.5328 = 57.6803$$
   This is the present value. The future value immediately after the last payment at time 39 (since the first payment was at time 0) would then be:
$$FV = PV(1.03)^{39} = 57.6803(1.03)^{39} = 182.6751$$

About the Practice Exams

After learning the material in each module, and after reviewing the modules and midterm exams in this manual, you will be ready to tackle these practice exams. Like the Society of Actuaries' Exam FM, each of these practice exams consists of 30 questions on the topics in the FM syllabus. As in SOA Exam FM, if your answer for a numerical question does not match any of the answer choices exactly, you should select the choice that is closest to your calculated answer.

The 14 practice exams fall into 3 categories:

a) The first 6 practice exams are relatively straightforward, to enable you to review the basics of each topic. You may want to attempt them in a *non-timed* environment to evaluate your skills and understanding.

b) The next 5 practice exams introduce more difficult questions in order to replicate the actual exam experience. You should take each of these in a *timed* environment, allowing yourself 2.5 hours to complete each exam. This will give you experience with exam-like conditions.

c) The final 3 practice exams include especially challenging problems. Try to finish these exams within 2.5 hours, but you may need extra time due to the difficulty of the problems. The important thing is to be sure you understand the solutions, so that you will be able to apply these methods when you take Exam FM.

Please keep in mind that the actual exam questions are confidential, and there is no guarantee that the questions you encounter on Exam FM will look exactly like those included in these practice exams.

Practice Exam 1

Questions

1. You are given the following yield curve:

Year	Spot Rate
1	5.5%
2	5.2%
3	5.0%
4	4.4%
5	4.0%

What is the 4-year forward rate ($i_{4,5}$)?

(A) 2.2% (B) 2.3% (C) 2.4% (D) 2.5% (E) 2.6%

2. Find the Macaulay duration of a 10-year, 1,000 par value bond with 8% annual coupons, based on an annual effective yield of 6.5%.

(A) 7.2 (B) 7.4 (C) 7.6 (D) 7.8 (E) 8.0

3. At an annual effective loan interest rate i, a loan of K can be repaid in either of two ways:

i) 475 now and 475 in 1 year, or

ii) 570 in 2 years and 570 in 3 years.

Calculate K.

(A) 893 (B) 901 (C) 909 (D) 917 (E) 925

4. A 10-year annuity-immediate pays 100 quarterly for the first year. In each subsequent year, the quarterly payment amount is increased by 5% over the payment amount during the previous year. At a nominal annual interest rate of 8% convertible quarterly, what is the present value of this annuity?

(A) 2,997 (B) 3,075 (C) 3,108 (D) 3,225 (E) 3,333

5. At an annual effective interest rate i, the present value of a 10-year annuity-immediate with level annual payments is X. At the same interest rate, a 20-year annuity-immediate with the same annual payment amount has a present value of $1.5X$. Calculate i.

(A) 7.2% (B) 7.4% (C) 7.6% (D) 7.8% (E) 8.0%

6. A 10-year bond that pays 4.3% annual coupons has a price equal to its face amount. (In other words, it is a "par bond.") A similar 10-year bond has the same yield rate and has a price (at issue) of 104 per 100 of face amount. What is this second bond's annual coupon rate?

(A) 4.4% (B) 4.5% (C) 4.6% (D) 4.7% (E) 4.8%

7. Spot rates for terms of 1 to 4 years are as follows:

Term (in years)	1	2	3	4
Spot Rate	5.0%	5.75%	6.25%	X

The coupon rate for a 4-year par bond is 6.62%. Calculate X.

(A) 6.60% (B) 6.65% (C) 6.70% (D) 6.75% (E) 6.80%

8. A company has liabilities requiring payments of 1,000; 3,000; and 5,000 at the end of years 1, 2 and 3, respectively. The investments available to the company are the following zero-coupon bonds:

Maturity (years)	Annual Effective Yield	Par Value
1	7%	1,000
2	8%	1,000
3	9%	1,000

Determine the cost to match the company's liability cash flows exactly.

(A) 6,918 (B) 7,024 (C) 7,165 (D) 7,368 (E) 7,522

9. Pete Moss creates a retirement fund by making deposits at the end of each month for 20 years. For the first 10 years he deposits 100 per month, and for the last 10 years he deposits 200 per month. The fund earns interest at a nominal annual rate of 6% convertible monthly. At the end of 20 years, Pete uses the proceeds to purchase a 30-year annuity-immediate with monthly payments. The annuity is priced based on a nominal rate of 8% convertible monthly. What is the amount of Pete's monthly payment from this annuity?

(A) 408 (B) 425 (C) 437 (D) 441 (E) 459

10. An annuity makes annual payments at the beginning of each year for 20 years. For the first 10 years the payments are 100. Starting with the 11$^{\text{th}}$ payment, each payment is increased by 6% over the previous payment. At an annual effective rate of 8%, what is the present value of this annuity?

(A) 1,177 (B) 1,190 (C) 1,202 (D) 1,213 (E) 1,225

11. An annual-coupon corporate bond has an annual effective yield of 7.2% at its current price of 972.48. At 7.2%, the bond's Macaulay duration is 7.1245. Using the first-order modified approximation method, estimate the change in price that would cause the bond's yield to increase by 0.10%.

(A) −6.463 (B) −6.685 (C) −6.814 (D) −7.012 (E) −7.163

12. A 40-year loan is repaid by level annual payments at the end of each year. The principal paid in the 20^{th} payment is 162.43 and the principal paid in the 25^{th} payment is 238.66. Find the amount that was borrowed.

(A) 9,500 (B) 9,750 (C) 10,000 (D) 10,250 (E) 10,500

13. Thelma deposits 100 into an account at the end of each year for 20 years. Her account earns interest at an annual effective rate of 5%. Louise deposits money into an account at the end of each year for 20 years. Her account also earns interest at an annual effective rate of 5%. The amount of Louise's deposit increases each year in the following pattern: $P, 2P, 3P, \ldots, 20P$. At the end of 20 years the balances in the two accounts are equal. Calculate P.

(A) 10.93 (B) 11.05 (C) 11.12 (D) 11.23 (E) 11.35

14. Dinah Soares borrows money to buy a new piano. She agrees to pay back the loan with level annual payments at the end of each year for 30 years. The annual effective interest rate is 7%. The amount of interest in her 10^{th} payment is 366.74. What is the amount of interest in her 20^{th} payment?

(A) 221.86 (B) 229.64 (C) 244.18 (D) 250.72 (E) 253.80

15. Rita Booke makes a deposit into an account. For the first 5 years the account accumulates at a force of interest of 0.05. For the next 10 years the fund accumulates at a nominal annual rate of discount of 6% convertible quarterly. For the 15-year period, what is the equivalent nominal annual interest rate convertible monthly?

(A) 5.59% (B) 5.71% (C) 5.83% (D) 5.96% (E) 6.04%

16. Robin Banks purchases a 10-year 1,000 par value bond that pays semi-annual coupons at an 8% annual rate. The bond is priced to yield 7.5% convertible semi-annually. Robin reinvests the coupon payments in a fund that pays a nominal rate of 7% convertible semi-annually. Over the 10-year period, what is Robin's nominal annual yield convertible semi-annually?

(A) 7.36% (B) 7.41% (C) 7.48% (D) 7.56% (E) 7.63%

17. You are given the following yield curve:

Year	Spot Rate
1	4.0%
2	4.2%
3	4.6%
4	—
5	5.1%

Given that the 4-year forward rate ($i_{4,5}$) is 6.1%, calculate the 4-year spot rate (s_4).

(A) 4.81% (B) 4.83% (C) 4.85% (D) 4.87% (E) 4.89%

18. A 20-year annuity-immediate has annual payments. The first payment is 100 and subsequent payments increase by 100 each year until they reach 1,000. Each of the remaining payments is 1,000. At an annual effective interest rate of 7.5%, what is the present value of this annuity?

(A) 6,201 (B) 6,372 (C) 6,413 (D) 6,584 (E) 6,700

19. Vera Phyde buys a 1,000 par value 5-year zero-coupon bond priced to yield a 6% annual effective interest rate. At the same time she buys a 1,000 par value 5-year bond with 8% semiannual coupons that is priced to yield 7% convertible semi-annually. The coupon payments are reinvested at 6.5% convertible semi-annually. Over their 5-year term, what is Vera's annual effective yield from the combination of these investments?

(A) 6.0% (B) 6.2% (C) 6.4% (D) 6.6% (E) 6.8%

20. Account A earns compound interest at an annual effective rate i (where i is greater than 0). Account B earns compound interest at an annual effective rate equal to $1.1 \cdot i$.

1,000 is deposited into each of these accounts at $t = 0$. No other deposits or withdrawals occur. At the end of 20 years (at $t = 20$), the balance in Account B will be 10% larger than the balance in the Account A.

What will be the difference between the balances in the two accounts at the end of 10 years (at $t = 10$)?

(A) 75 (B) 80 (C) 85 (D) 90 (E) 95

21. A 20-year monthly-payment variable-rate mortgage has an initial principal of 200,000 and an initial interest rate of 3.6% convertible monthly.

At the end of 2 years, the mortgage interest rate increases from 3.6% to 4.5% and a new monthly payment amount (for the 3^{rd} and later years) is calculated, based on the new interest rate and the outstanding loan balance on that date. There are no other interest rate changes during the first 5 years of the loan.

To the nearest 100, what is the outstanding balance of the loan at the end of 5 years (immediately after the 60^{th} monthly payment)?

(A) 162,600 (B) 162,900 (C) 164,000 (D) 164,300 (E) 165,400

22. A 20-year bond has a face value (and maturity value) of 1,000. It pays semi-annual coupons at a 6% (annual) coupon rate. The bond is callable on any coupon date on or after its 10^{th} anniversary, with a 5% call premium.

If an investor purchases this bond on its issue date at a price of 1,060, and holds the bond until it matures or is called, what is the minimum yield the investor could earn (expressed as a nominal rate, convertible semi-annually)?

(A) 5.50% (B) 5.59% (C) 5.76% (D) 5.96% (E) 6.37%

23. Amanda makes a deposit of 1,000 at the end of each month for 30 years into an account that earns an annual effective interest rate i. Beginning one month after the last deposit, she can make perpetual monthly withdrawals of 4,000 from the same account at the same interest rate.

Calculate i.

(A) 5.3% (B) 5.5% (C) 5.7% (D) 5.8% (E) 6.0%

24. Charlotte takes out a 30-year mortgage for an amount A at an interest rate i convertible monthly. She notices that if she makes all the monthly payments for 30 years, the total amount of her payments will equal 3 times the amount she borrowed. What percentage of the loan principal will Charlotte repay during the first 2 years of the loan?

(A) 1.32% (B) 1.38% (C) 1.42% (D) 1.46% (E) 1.49%

25. Barb Dwyer invests 500 at $t = 0$ at a nominal annual interest rate of 6% convertible quarterly. What additional amount will Barb need to invest at $t = 2$ in order to have a total of 1,000 at $t = 5$?

(A) 242 (B) 273 (C) 278 (D) 290 (E) 327

26. Lisa Carr purchased a newly-issued 25-year bond that pays semi-annual coupons at an 8% (annual) rate. The bond has a par value (and a redemption value) of 1,000. The bond is callable on or after its 10$^{\text{th}}$ anniversary with a 10% call premium (i.e., 1,100 is payable if it is called).

    Lisa purchased this bond at issue at a price that will assure her a rate of return of at least 7.5% (a nominal rate, convertible semi-annually). If the bond is called on its 12$^{\text{th}}$ anniversary, what is Lisa's actual rate of return?

    (A) 7.5% (B) 7.6% (C) 7.7% (D) 7.8% (E) 7.9%

27. A 20-year bond has a par value of 1,000 and a maturity value of 1,040. It pays semi-annual coupons at a 5% annual rate. Calculate the bond's purchase price if its yield to maturity is 6% convertible monthly.

    (A) 876 (B) 880 (C) 884 (D) 889 (E) 897

28. The following table gives the term structure of spot interest rates.

Term (in years)	1	2	3	4
Spot interest rate	4.25%	4.80%	X	5.40%

    If the two-year forward rate ($i_{2,3}$) is 6.5%, what is the value of X?

    (A) 4.95% (B) 5.04% (C) 5.20% (D) 5.36% (E) 11.61%

29. Sarah Naide takes out a loan for 50,000 with 30 quarterly payments. For the first 10 payments, Sarah will pay only the interest due at the end of each quarter. For the last 20 payments, Sarah will pay X at the end of each quarter (an amount that will pay off the loan). If the annual effective interest rate for the loan is 5.8%, what is the total of all of Sarah's payments for this loan?

    (A) 50,709 (B) 57,785 (C) 64,882 (D) 65,330 (E) 67,050

30. On January 1, Patty O. opens a savings account. She deposits 300 at the beginning of odd months (January, March, etc.) and withdraws 250 at the beginning of even months (February, April, etc.). If Patty earns an interest rate of 6.3% convertible monthly, what is the value of her fund at the end of 12 months? (Assume 30-day months.)

    (A) 310 (B) 319 (C) 328 (D) 337 (E) 349

Auto-Grade My Exam (Click Here)

1. The four-year forward rate ($i_{4,5}$) is given by:

$$1{+}i_{4,5} = (1+s_5)^5/(1+s_4)^4 = 1.04^5/1.044^4 = 1.02415$$
$$i_{4,5} = 0.024$$

(*Note the unusually low value, due to the inverted yield curve.*)

Answer C

2. $D_{\text{mac}} = \dfrac{80(Ia)_{\overline{10|}} + 10 \times 1,000 \cdot v^{10}}{\text{Bond Price}}$

$v^{10} = 1.065^{-10} = 0.532726 \qquad \ddot{a}_{\overline{10|}} = 7.6561 \qquad a_{\overline{10|}} = 7.1888$

$(Ia)_{\overline{10|}} = \frac{\ddot{a}_{\overline{10|}} - 10 \cdot v^{10}}{i} = 35.8284$

Bond Price $= 80 \cdot a_{\overline{10|}} + 1,000 \cdot v^{10} = 1,107.83$

Or set N = 10, PMT = 80, I/Y =6.5, FV = 1,000 and CPT PV = −1,107.83.

$D_{\text{mac}} = (80 \times 35.8284 + 10,000 \times 0.532726)/1,107.83 = 7.396$

Answer B

3. $K = 475 + 475 \cdot v = 570 \cdot v^2 + 570 \cdot v^3 = v^2 \cdot (570 + 570 \cdot v)$

$475 \cdot (1+v) = v^2 \cdot 570 \cdot (1+v) \to v^2 = 475/570 = 0.8333$

$v = 0.8333^{0.5} = 0.91287$

Note that each payment of 570 occurs 2 years after each payment of 475, so we could have written: $\quad 570v^2 = 475 \qquad v = (475/570)^{0.5} = 0.91287$

$K = 475(1+v) = 475(1.91287) = 908.61$

Answer C

4. The accumulated value of the first year's payments at the end of year 1 is 412.16.
(Set N = 4, I/Y = 2, PMT = 100, PV =0. CPT FV = −412.16.)

Thus the value of the annuity is the same as that of a 10-year annuity-immediate with annual payments, the first being 412.16 and subsequent payments increasing by 5% each year. The annual effective interest rate is:

$$i = (1.02)^4 - 1 = 0.08243$$

The present value of this annuity is:

$$412.16 \cdot a^{5\%}_{\overline{10|}8.2432\%} = 412.16 \cdot \frac{1 - (1.05/1.08243)^{10}}{0.08243 - 0.05} = 3,333.28$$