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2nd Edition

Abraham Weishaus, Ph.D., FSA, CFA, MAAA

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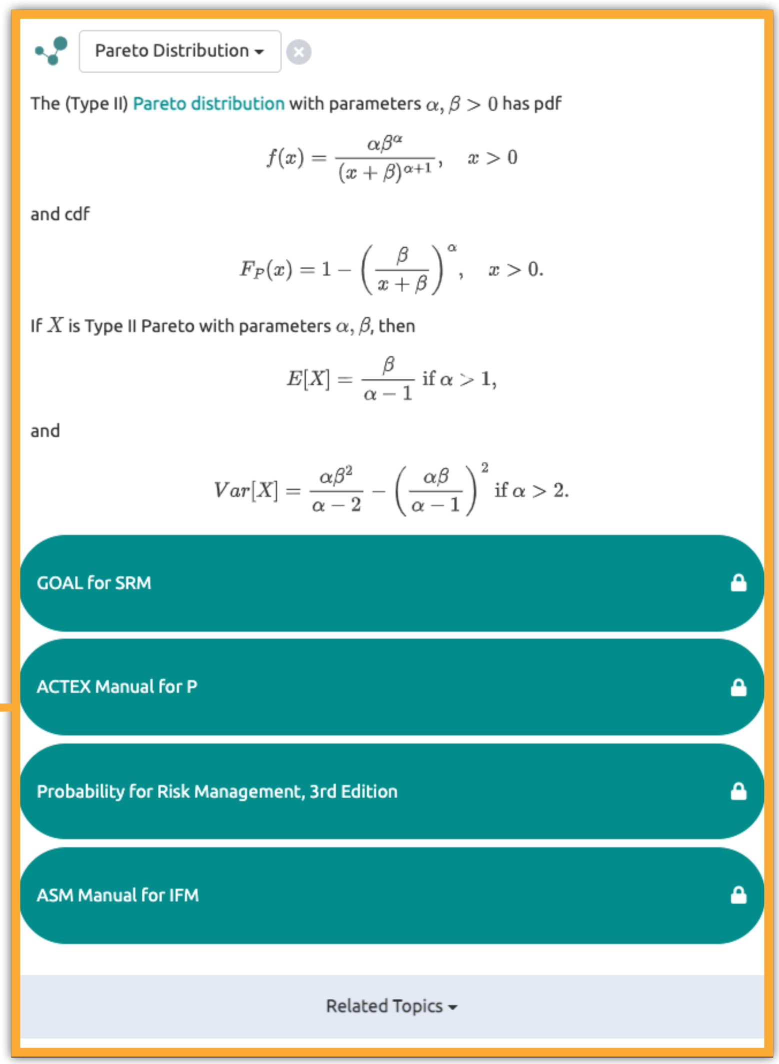

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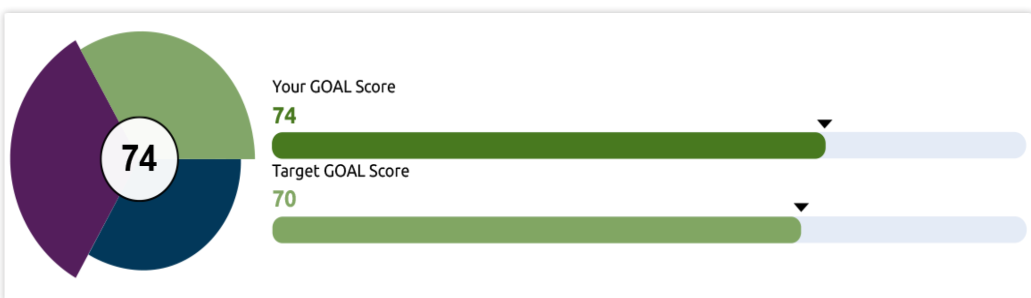

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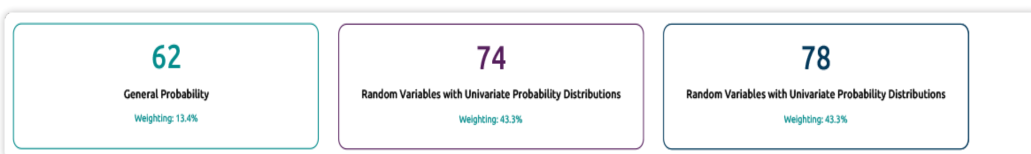

See key areas where you can improve.

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Advanced	60	169 / 304
Mastery	78	43 / 304

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Created	Last Accessed	Completed	Mode	Categories	Questions	Status	Status
05/24/2022 21:57:43	05/24/2022 21:57:43		Quiz	Continuous P...	25	New	Resume
05/24/2022 10:34:05	05/24/2022 14:57:49	05/24/2022 14:57:49	Practice Session	Addition and ...	80	Complete	Review
05/21/2022 19:32:50	05/23/2022 20:02:00		Simulated Exam	Exam 2	30	Reviewing	Complete
05/17/2022 15:19:19	05/17/2022 15:46:03	05/23/2022 14:15:29	Simulated Exam	Exam 6	30	Complete	Review
05/14/2022 11:26:59	05/14/2020 12:02:36	05/23/2022 11:57:47	Quiz	Conditional D...	20	Complete	Review

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Preface

Welcome to Exam ALTAM!

On Exam FAM-L, you learned the basics of life insurance mathematics; mortality models, insurances, annuities, premiums, and reserves. On ALTAM, we discuss more general long-term insurances. Sometimes insurance covers more than one life. Sometimes the benefit depends on how one died. And there are other long-term products to analyze, such as disability income and long term care. We discuss all of these products, and also discuss pricing and analysis of profit.

Syllabus

According to the syllabus, the exam is 3 hours and will consist of 60 points of written answer questions. The topics this exam will cover are:

1. Multistate models
2. Multiple decrement models
3. Multiple life models
4. Pensions
5. Profit tests

The textbook for the course is *Actuarial Mathematics for Life Contingent Risks* third edition. This is a college-style textbook. It is oriented towards practical application rather than exam preparation. Almost all exercises require use of spreadsheets or derivation of formulas. The syllabus also includes a study note on variable annuities.

The syllabus splits the material into 7 topics, with the following weights:

Topic	Weight	Lessons in this manual
Multiple State Survival Models	10–20%	3–4, 7–12
Multiple State Insurances and Annuities	12–20%	2, 5–6, 13
Joint Life Insurance and Annuities	8–16%	15–22
Profit Analysis	10–20%	27–28
Pension Plans and Retirement Benefits	10–18%	24–26
Universal Life Insurance	10–18%	29
Embedded Options in Life Insurance and Annuity Products	10–18%	30–35

Based on the Spring 2023 exam, it appears that each exam will have six questions, each one about 10 points. Thus at least one of the topics will not appear, unless a single question spans two topics. On the Spring 2023 exam, the two Markov chain questions spanned the first two topics in the list above, and the UL question had some Profit Analysis in it. The other topics were covered by one question apiece.

Here is the distribution of number of questions by topic on MLC and LTAM exams, based on the organization of this manual, for the Spring 2012 through Fall 2013 exams, and the points per topic for the 2014 and later exams. Note that each question is classified based on the highest lesson required for it, so a question involving an asset share on universal life (there was one such question) would be classified as a universal life question. (Asset shares are not on the current ALTAM syllabus.) Thus a 0 does not indicate no questions on the topic on the exam. For example, the Fall 2016 exam had a question involving interest rate models (a topic not on the ALTAM syllabus) applied to a multiple-life insurance, even though the table shows 0 for multiple life models.

		Questions				Points																	
		MLC				MLC									LTAM								
Topic	Lessons	Spr 12	Fall 12	Spr 13	Fall 13	Spr 14	Fall 14	Spr 15	Fall 15	Spr 16	Fall 16	Spr 17	Fall 17	Spr 18	Fall 18	Spr 19	Fall 19	Spr 20	Fall 20	Spr 21	Fall 21A	Fall 21B	Spr 22
Thiele's equation	2	1	1	0	0	0	0	0	0	0	0	2	0	2	0	0	0	0	2	0	0	2	0
Markov chains	3–6	3	3	3	3	9	12	15	6	16	7	14	4	13	14	16	21	18	6	22	15	15	18
Multi-decrement models	8–14	2	1	2	0	2	2	6	8	0	9	4	11	0	2	2	6	12	4	2	2	6	2
Multiple life models	15–23	2	2	2	1	7	8	2	11	2	0	9	9	12	4	13	10	12	14	27	9	5	12
Retirement benefits	24–26	1	0	1	1	11	11	8	6	15	14	13	11	13	14	12	4	12	12	12	14	10	13
Profit tests	27–29	5	4	3	2	20	26	13	26	18	16	20	10	6	6	8	9	8	4	2	17	16	12
Total		13	11	11	7	49	59	44	57	51	46	62	45	46	40	51	48	62	42	40	57	54	57

In this table, only topics on the ALTAM syllabus are included, so total questions and points are not the total points on the exam. Total was 30 questions for Spring 2012, 25 questions for Fall 2012–Fall 2013, 96 points for all exams Spring 2014 and later. Also, Lessons 30–35 are new for ALTAM. And LTAM did not have Universal Life in its syllabus, although MLC did.

As you can see in this table, weights on the topics have varied. Weights can vary significantly since written answer questions have heavy weights and cover different topics in different years.

Other downloads from the SOA site

The introductory study note is at

https://www.soa.org/globalassets/assets/files/edu/2023/2023-07-altam-notation-term.pdf

You will receive the life tables as a worksheet at the exam. It is at

https://www.soa.org/globalassets/assets/files/edu/2023/2023-07-altam-excel-workbook-tables.xlsx

You may use this worksheet to obtain normal functions if needed. You may also use the worksheet to do your calculations, as long as you explain what you did. This worksheet will have most of the functions of Excel, but will not have VBA or Solver. It appears that you will not be asked questions requiring heavy mathematical calculation that can only be done on a spreadsheet.

Another set of tables, the tables from the former MLC, will be useful if you wish to work on old exam questions that use the Illustrative Life Table. You can find them at

https://www.soa.org/Files/Edu/edu-2013-mlc-tables.pdf

However, I have converted all pre-2012 exam questions to use the Standard Ultimate Life Table, and the SOA converted questions from 2012 and later when they incorporated them in their sample questions. So it is unlikely you'll need the Illustrative Life Table.

In addition to the tables, you will get the following formula sheet:

https://www.soa.org/globalassets/assets/files/edu/2023/2023-07-altam-formula-sheet.pdf

Thus you need not memorize Woolhouse's formula or the Black-Scholes put option formula.

The notation and terminology study note is at

https://www.soa.org/globalassets/assets/files/edu/2023/2023-07-altam-notation-term.pdf

This manual follows the conventions of this note, but after you've finished the manual, you may want to read through this note.

Sample questions

The SOA's sample questions are at

https://www.soa.org/globalassets/assets/files/edu/2023/spring/2023-07-altam-sample-questions.pdf

and their solutions are at

https://www.soa.org/globalassets/assets/files/edu/2023/spring/2023-07-altam-sample-solutions.pdf

These questions are mostly from old exams, but some have been modified. This manual has solutions to old exam questions in Appendix B, but those solutions are to unmodified questions. For questions that used the Illustrative Life Tables, those solutions would be different. But for questions that merely modified terminology (e.g. replacing "reserve" with "policy value") they may be the same.

At the end of each lesson there is a list of relevant sample questions. Most sample questions cover several lessons; the reference to the sample question is at the end of the last lesson covered by the question.

Exercises and old exam questions in this manual

There are about 440 original exercises in the manual and about 260 old exam questions. Even though the exam is written answer, I have retained these short answer and multiple choice questions. You will get a lot of good practice doing these questions; they are not a waste of time, even though written answer questions are less tricky and can potentially ask you for things that multiple choice questions can't, like deriving formulas.

The old exam questions come from old Course 150, 2000-syllabus Exam 3, Exam M, and Exam MLC. However, very few questions from the 2012 and later MLC exams are given in the exercises, so you may use those exams or the SOA sample questions as final practice. I've left references to the LTAM sample questions at the ends of the lessons.

SOA Course 150 from 1987 through 1991 had multiple choice questions in the morning and written answer questions in the afternoon. Since ALTAM consists of written answer questions, I've included all applicable written answer questions in the exercises.

Back in 1999, the CAS and SOA created a sample exam for the then-new 2000 syllabus. This exam had some questions from previous exams but also some new questions, some of them not multiple choice. This sample exam was never a real exam, and some of its questions were defective. This sample exam is no longer available on the web. I have included appropriate questions from it. *Whenever an exercise is labeled 1999 C3 Sample, it refers to the 1999 sample, not the current list of sample questions.*

Questions from old exams are marked xxx:yy, where xxx is the time the exam was given, with S for spring and F for fall followed by a 2-digit year, and yy is the question number. There was a period in the 1990s when the SOA, while it allowed use of its old exam questions, did not want people to reveal which exam they came from. As a result, I sometimes had study notes for old exams in this period and could not identify the exam they came from. In such a case, I mark the question aaa-bb-cc:yy, where aaa-bb-cc is the study note number and yy is the question number. Generally aaa is the exam number (like 150), and cc is the 2-digit year the study note was published.

Study schedule

Although this manual is large, much of it is exercises and practice exams. You do not have to do every exercise; do enough to gain confidence with the material. With intense studying, you should be able to cover all the material in 3 months.

It is up to you to set up a study schedule. Different students will have different speeds and different constraints, so it's hard to create a study schedule useful for everybody. However, I offer a sample 10.5-week study schedule, Table 1, as a guide. This study schedule omits the first lesson. which is review.

The study schedule lists lessons that are either long or hard, as well as those that are short or easy or just background, so that you may better allocate your study time within the study periods provided for each subject.

Table 1: 10.5 Week Study Schedule for Exam ALTAM

Subject	Lessons	Study Period	Hard/Long Lessons	Easy/Short Lessons
Thiele's Equation	2	1 day		
Markov Chains	3–7	2 weeks	6	
Multiple Decrements	8–13	1 week	11	10
Multiple Lives	15–22	1.5 weeks		15,18
Pension	24–26	2 weeks		
Profit Tests—Traditional	27–28	1 week		28
Universal Life	29	0.5 weeks		
Variable Life	30–35	1.5 weeks		

Acknowledgements

I would like to thank the SOA and CAS for allowing me to use their old exam questions. I'd also like to thank Harold Cherry for suggesting this manual and for providing three of the pre-2000 SOA exams and all of the pre-2000 CAS exams I used.

The creators of TEX, LATEX, and its multitude of packages all deserve thanks for making possible the professional typesetting of this mathematical material.

I would like to thank all readers who sent in errata. Here is a partial list: Steve Guo, Professor Wafaa Shaban.

Errata

Please report all errors you find in these notes to the author. You may send them to the publisher at mail@studymanuals.com or directly to me at errata@aceyourexams.net. Please identify the manual and edition the error is in. This is the 2nd edition of the SOA Exam ALTAM manual.

An errata list will be posted at errata.aceyourexams.net. Check this errata list frequently.

Flashcards

Many students find flashcards a useful tool for learning key formulas and concepts. ASM flashcards, available from the same distributors that sell this manual, contain the formulas and concepts from this manual in a convenient deck of cards. The cards have cross references, usually by page, to the manual.

Lesson 2

Thiele's differential equation

Reading: *Actuarial Mathematics for Life Contingent Risks* 3rd edition 7.4

Thiele's differential equation is a version of the policy value recursion for fully continuous insurances or annuities[1] with continuous premiums. The formula is

Thiele's differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_tV = \delta_t\,{}_tV + G_t - e_t - (b_t + E_t - {}_tV)\mu_{[x]+t} \tag{2.1}$$

A proof is provided in a sidebar. Know this equation well, since on exams you may be asked to state it. Intuitively, the equation says that the policy value increases at the rate of interest, and with premiums minus expenses, minus mortality benefits based on the net amount at risk.

Usually the equation doesn't have a closed form solution. It can be solved numerically using Euler's method. Namely, the derivative is the limit as $h \to 0$ of a difference quotient:

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_tV = \lim_{h\to 0}\frac{{}_{t+h}V - {}_tV}{h}$$

Euler's numerical method uses the difference quotient as an approximation to the derivative in Thiele's equation. In other words,

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_tV \approx \frac{{}_{t+h}V - {}_tV}{h}$$

Plug in this expression for the derivative into equation (2.1) to obtain

$${}_{t+h}V - {}_tV \approx h\big(\delta_t\,{}_tV + G_t - e_t - (b_t + E_t - {}_tV)\mu_{[x]+t}\big) \tag{2.2}$$

Notice that this formula is slightly different from the discrete counterpart, equation (1.57), in that the net amount of risk is the face amount minus the *beginning* policy value, whereas in the discrete case the *end-of-year* policy value was subtracted from the face amount. In the continuous case, beginning of period and end of period are the same, but in the approximation the period length is h so they're not quite the same. This causes a slight difference in the resulting recursive formula.

Solving for ${}_tV$, we get the following equation

$${}_tV = \frac{{}_{t+h}V - h\big(G_t - e_t - (b_t + E_t)\mu_{[x]+t}\big)}{1 + h(\mu_{[x]+t} + \delta)} \tag{2.3}$$

The way we have done it is the way *Actuarial Mathematics for Life Contingent Risks* works out its examples. We use the derivative at time t to relate ${}_tV$ to ${}_{t+h}V$, or equivalently, we use the derivative at *time $t-h$* to relate ${}_{t-h}V$ to ${}_tV$. This method is called the *Forward Euler Approximation*, since it expresses the derivative at time t in terms of values at times t and $t+h$. An alternative method would be to use the derivative at *time t* to relate ${}_{t-h}V$ to ${}_tV$. In other words, start with the following approximation:

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_tV \approx \frac{{}_tV - {}_{t-h}V}{h}$$

[1] *Actuarial Mathematics for Life Contingent Risks* does not apply the formula to annuities, but it could be applied by treating the annuity payments as negative premiums and setting the face amount equal to zero.

Proof of Thiele's Differential Equation

Let π_t be the net premium at time t. It suffices to prove the equation for net premium policy values, since if it is true for net premium policy values, replacing π_t with $G_t - e_t$ and b_t with $b_t + E_t$ yields the gross premium policy value formula. Let $v(t)$ be the discount factor for cash flows at time t:

$$v(t) = \exp\left(-\int_0^t \delta_t \,\mathrm{d}t\right)$$

The policy value at time t is the expected present value of future benefits minus the expected present value of future premiums, or

$${}_tV = \int_0^\infty b_{t+u}\frac{v(t+u)}{v(t)}\,{}_up_{x+t}\,\mu_{x+t+u}\,\mathrm{d}u - \int_0^\infty \pi_{t+u}\frac{v(t+u)}{v(t)}\,{}_up_{x+t}\,\mathrm{d}u$$

and setting $s = t + u$ and using ${}_up_{x+t} = {}_{t+u}p_x/{}_tp_x$,

$$\begin{aligned}{}_tV &= \frac{1}{v(t)\,{}_tp_x}\left(\int_t^\infty b_s\,v(s)\,{}_sp_x\,\mu_{x+s}\,\mathrm{d}s - \int_t^\infty \pi_s\,v(s)\,{}_sp_x\,\mathrm{d}s\right)\\ v(t)\,{}_tp_x\,{}_tV &= \int_t^\infty b_s\,v(s)\,{}_sp_x\,\mu_{x+s}\,\mathrm{d}s - \int_t^\infty \pi_s\,v(s)\,{}_sp_x\,\mathrm{d}s \qquad (*)\end{aligned}$$

We differentiate both sides with respect to t. On the right side, the derivative of an integral with t as the lower bound is negative the integrand, so we get

$$v(t)\,{}_tp_x(\pi_t - b_t\,\mu_{x+t}) \qquad (**)$$

To differentiate the left side, a product of three factors, it is easier to use logarithmic differentiation. Remember that

$$\frac{\mathrm{d}\ln f(t)}{\mathrm{d}t} = \frac{\mathrm{d}f(t)/\mathrm{d}t}{f(t)}$$

so

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = f(t)\frac{\mathrm{d}\ln f(t)}{\mathrm{d}t}$$

The logarithm of the left side is

$$\begin{aligned}\ln v(t) + \ln {}_tp_x + \ln {}_tV &= \ln\exp\left(-\int_0^t \delta_u\,\mathrm{d}u\right) + \ln\exp\left(-\int_0^t \mu_{x+u}\,\mathrm{d}u\right) + \ln {}_tV\\ &= -\int_0^t (\delta_u + \mu_{x+u})\mathrm{d}u + \ln {}_tV\end{aligned}$$

The derivative of an integral with t at the upper bound is the integrand, so the derivative equals

$$-\delta_t - \mu_{x+t} + \frac{\mathrm{d}\ln {}_tV}{\mathrm{d}t}$$

and by logarithmic differentiation,

$$\frac{\mathrm{d}\ln {}_tV}{\mathrm{d}t} = \frac{\mathrm{d}_tV/\mathrm{d}t}{{}_tV}$$

The derivative of the left side of (*) is therefore

$$v(t)\,{}_tp_x\,{}_tV\left(-\delta_t - \mu_{x+t} + \frac{\mathrm{d}_tV/\mathrm{d}t}{{}_tV}\right) = v(t)\,{}_tp_x\left(\frac{\mathrm{d}_tV}{\mathrm{d}t} - {}_tV(\mu_{x+t} + \delta_t)\right)$$

Equating this to (**) and canceling $v(t)\,{}_tp_x$ from both sides:

$$\frac{\mathrm{d}_tV}{\mathrm{d}t} - {}_tV(\mu_{x+t} + \delta_t) = \pi_t - b_t\,\mu_{x+t}$$
$$\frac{\mathrm{d}_tV}{\mathrm{d}t} = \delta_t\,{}_tV + \pi_t - \mu_{x+t}(b_t - {}_tV)$$

which is Thiele's equation.

This approximation is called the ***Backward Euler Approximation***, since it expresses the derivative at time t in terms of values at times t and $t-h$. Plugging this into equation (2.1)

$$ {}_tV - {}_{t-h}V \approx h\big(\delta_t\,{}_tV + G_t - e_t - (b_t + E_t - {}_tV)\mu_{[x]+t}\big) \tag{2.4}$$

Solving for ${}_{t-h}V$, we get

$$ {}_{t-h}V = {}_tV\big(1 - h(\mu_{[x]+t} + \delta)\big) + h\big(-G_t + e_t + (b_t + E_t)\mu_{[x]+t}\big) \tag{2.5}$$

To use equations (2.3) or (2.5), start with the maturity/expiry date. At that time, the policy value is 0 for term insurance and the pure endowment value for endowment insurance; whole life can be treated as an endowment insurance maturing at a high age. Then work backwards, using a small value of h.

For net premium policy values, the expense terms would be omitted and G_t would be the net premium.

The two alternative equations lead to different results. Which one should be used? The exam will state which derivatives to use. Notice that equation (2.3) uses the derivative at time t to go from $t+h$ to t; in other words, the derivative at the *bottom* of the interval is used. On the other hand, equation (2.5) uses the derivative at time t to go from t to $t-h$; in other words, the derivative at the *top* of the interval is used.

If it is not stated which derivatives to use, use equation (2.3), since that's what the textbook uses.

Example 2A For a fully continuous whole life insurance of 1000 on (50), you are given

(i) Mortality is uniformly distributed with $\omega = 100$.
(ii) $\delta = 0.05$

Calculate the net premium policy value at times 25 and 26. Then use the Euler approximation of Thiele's equation with steps of $h = 0.25$ to calculate the net premium policy value at time 25 from the one at 26. Do this two ways:

1. Using derivatives at 25.75, 25.5, 25.25, and 25.
2. Using derivatives at 26, 25.75, 25.5, and 25.25.

Solution: The exact policy values, computed with the insurance ratio formula, are

$$\bar{A}_{50} = \frac{1 - e^{-0.05(50)}}{0.05(50)} = 0.367166$$
$$\bar{A}_{75} = \frac{1 - e^{-0.05(25)}}{0.05(25)} = 0.570796$$
$$\bar{A}_{76} = \frac{1 - e^{-0.05(24)}}{0.05(24)} = 0.582338$$
$${}_{25}V = 1000\left(\frac{0.570796 - 0.367166}{1 - 0.367166}\right) = \boxed{\mathbf{321.775}}$$
$${}_{26}V = 1000\left(\frac{0.582338 - 0.367166}{1 - 0.367166}\right) = \boxed{\mathbf{340.014}}$$

For the approximation, the premium is

$$P = 1000\left(\frac{\delta\bar{A}_{50}}{1 - \bar{A}_{50}}\right) = 29.010$$

and the μ's are $1/(50-x)$ for $x = 25, 25.25, 25.5, 25.75$. To use derivatives at 25.75, 25.5, 25.25, and 25, we use equation (2.3),

$$\begin{aligned}
{}_{25.75}V &= \frac{340.014 - 0.25\bigl(29.010 - 1000/24.25\bigr)}{1 + 0.25(0.05 + 1/24.25)} = 335.4201\\
{}_{25.5}V &= \frac{335.4201 - 0.25\bigl(29.010 - 1000/24.5\bigr)}{1 + 0.25(0.05 + 1/24.5)} = 330.8598\\
{}_{25.25}V &= \frac{330.8598 - 0.25\bigl(29.010 - 1000/24.75\bigr)}{1 + 0.25(0.05 + 1/24.75)} = 326.3329\\
{}_{25}V &= \frac{326.3329 - 0.25\bigl(29.010 - 1000/25\bigr)}{1 + 0.25(0.05 + 1/25)} = \boxed{\mathbf{321.839}}
\end{aligned}$$

which is about 6 cents too high. Greater accuracy could be achieved by using a smaller step.

To use derivatives at 26, 25.75, 25.5, and 25.25, we use equation (2.5).

$$\begin{aligned}
{}_{25.75}V &= 340.014\bigl(1 - 0.25(1/24 + 0.05)\bigr) + 0.25(-29.010 + 1000/24) = 335.3859\\
{}_{25.5}V &= 335.3859\bigl(1 - 0.25(1/24.25 + 0.05)\bigr) + 0.25(-29.010 + 1000/24.25) = 330.7928\\
{}_{25.25}V &= 330.7928\bigl(1 - 0.25(1/24.5 + 0.05)\bigr) + 0.25(-29.010 + 1000/24.5) = 326.2341\\
{}_{25}V &= 326.2341\bigl(1 - 0.25(1/24.75 + 0.05)\bigr) + 0.25(-29.010 + 1000/24.75) = \boxed{\mathbf{321.710}}
\end{aligned}$$

In this case, the answer is about 7 cents too low. □

Table 2.1: Summary of formulas and concepts in this lesson

Thiele's differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_tV = \delta_t\,{}_tV + G_t - e_t - (b_t + E_t - {}_tV)\mu_{[x]+t} \tag{2.1}$$

Numerical solutions with Euler's method:

Using derivatives at the lower end of each interval to go from $t+h$ to t:

$${}_{t+h}V - {}_tV \approx h\bigl(\delta_t\,{}_tV + G_t - e_t - (b_t + E_t - {}_tV)\mu_{[x]+t}\bigr) \tag{2.2}$$

$${}_tV = \frac{{}_{t+h}V - h\bigl(G_t - e_t - (b_t + E_t)\mu_{[x]+t}\bigr)}{1 + h(\mu_{[x]+t} + \delta)} \tag{2.3}$$

Using derivatives at the upper end of each interval to go from t to $t-h$:

$${}_tV - {}_{t-h}V \approx h\bigl(\delta_t\,{}_tV + G_t - e_t - (b_t + E_t - {}_tV)\mu_{[x]+t}\bigr) \tag{2.4}$$

$${}_{t-h}V = {}_tV\bigl(1 - h(\mu_{[x]+t} + \delta)\bigr) + h\bigl(-G_t + e_t + (b_t + E_t)\mu_{[x]+t}\bigr) \tag{2.5}$$

Exercises

2.1. **[150-F88:A5]**(6 points) Z is the present-value random variable for a whole life insurance of 1 payable at the moment of death of (x).

(a) Write an expression for Z.

(b) Using $\mathbf{E}[Z] = \mathbf{E}[Z \mid T \le h]\Pr(T \le h) + \mathbf{E}[Z \mid T > h]\Pr(T > h)$, demonstrate that

$$\bar{A}_x = \left(\int_0^h \frac{v^t\,{}_tp_x\,\mu_{x+t}\,\mathrm{d}t}{{}_hq_x}\right){}_hq_x + v^h\,{}_hp_x\,\bar{A}_{x+h}$$

(c) Using

$$\frac{\mathrm{d}\bar{A}_x}{\mathrm{d}x} = \lim_{h\to 0}\frac{\bar{A}_{x+h} - \bar{A}_x}{h}$$

demonstrate that

$$\frac{\mathrm{d}\bar{A}_x}{\mathrm{d}x} = (\mu_x + \delta)\bar{A}_x - \mu_x$$

2.2. For a fully continuous whole life insurance of 1000 on (40):

(i) Gross premiums are paid at a rate of 20 per year.
(ii) Initial expenses are 70% of premium plus 5
(iii) Expenses payable every year including the first are 6% of premium plus 0.4, payable continuously.
(iv) Settlement expenses are 15.
(v) $\delta = 0.04$
(vi) $\mu_{50} = 0.005$
(vii) ${}_{10}V^g = 122$

Calculate the derivative of the gross premium policy value at time 10.

2.3. For a fully continuous 10-year deferred whole life insurance on (55) of 100,000:

(i) Net premiums are payable for the first 10 years only.
(ii) Net premiums are 4800.
(iii) ${}_5V = 32{,}000$
(iv) $\frac{\mathrm{d}}{\mathrm{d}t}{}_tV = 6240$ at time 5.
(v) $\mu_{60} = 0.0032$

Determine δ.

2.4. For a fully continuous 30-year term insurance on (25) of 100,000:

(i) Net premiums are paid at a rate of 700 per year.
(ii) $\delta = 0.06$
(iii) $\mu_{40} = 0.008$
(iv) $\frac{\mathrm{d}}{\mathrm{d}t}{}_tV = -4.80$ at time 15.

Determine the net premium policy value at time 15.

Exercises continue on the next page . . .

2.5. For a fully continuous 20-year endowment insurance of 1000 on (45):

(i) The annual gross premium is 40.
(ii) Expenses are 5% of the gross premium.
(iii) $\mu_x = 0.002(1.01^x)$
(iv) $\delta = 0.05$

Using Euler's method with step 0.1 to numerically solve Thiele's differential equation, with derivatives at times 19.9 and 19.8, approximate ${}_{19.8}V$.

2.6. For a fully continuous 10-year term insurance of 100,000 on (55):

(i) The annual gross premium is 250.
(ii) Expenses are 3% of the gross premium, plus settlement expenses of 100.
(iii) $\mu_x = 0.001(1.015^x)$
(iv) $\delta = 0.04$

Using Euler's method with step 0.1 to numerically solve Thiele's differential equation, with derivatives at times 9.9 and 9.8, approximate ${}_{9.8}V$.

2.7. For a fully continuous 20-year deferred whole life insurance of 10,000 on (45):

(i) $\bar{A}_{65} = 0.25821$
(ii) The annual net premium is 71.25, and is payable for 20 years.
(iii) $\mu_x = 0.00015(1.06^x)$
(iv) $\delta = 0.05$

Using Euler's method with step 0.5 to numerically solve Thiele's differential equation, with derivatives at times 19.5 and 19, approximate ${}_{19}V$.

2.8. For a fully continuous 20-year endowment insurance of 10,000 on (45):

(i) The annual net premium is 359.76.
(ii) $\mu_x = 0.0002(1.065^x)$
(iii) $\delta = 0.04$

Using Euler's method with step 0.5 to numerically solve Thiele's differential equation, with derivatives at times 20 and 19.5, approximate ${}_{19}V$.

Exercises continue on the next page . . .

2.9. For a special 20-year term life insurance of 10,000 on (40), you are given:

(i) The death benefit is payable at the moment of death.
(ii) During the 5^{th} year the gross premium is 150 paid continuously at a constant rate
(iii) The force of mortality follows Gompertz's law $\mu_x = Bc^x$ with $B = 0.00004$ and $c = 1.1$.
(iv) The force of interest is 4%.
(v) Expenses are:
- 5% of premium payable continuously
- 100 payable at the moment of death

(vi) At the end of the 5^{th} year the expected value of the present value of future losses random variable is 1000.

Euler's method with steps of $h = 0.25$ years is used to calculate a numerical solution to Thiele's differential equation.

Calculate the expected value of the present value of future losses random variable at the end of 4.5 years.

(A) 975 (B) 962 (C) 949 (D) 936 (E) 923

2.10. For a 5-year warranty on Kira's new cell phone, you are given:

(i) The warranty pays 100 at the moment of breakage, if the phone breaks. The warranty only pays for one breakage.
(ii) If the phone has not broken, the warranty pays 100 at the end of 5 years.
(iii) Premiums of G are payable continuously at an annual rate of 25 until the phone breaks.
(iv) The force of breakage for this phone is $\mu_t = 0.02t$, $t \geq 0$.
(v) $\delta = 0.05$
(vi) ${}_tV$ denotes the gross premium policy value at time t for this warranty.
(vii) At the end of year 4, Kira's cell phone has not broken.
(viii) You approximate ${}_4V$ using Euler's method, with step size $h = 0.5$ and using the derivatives of ${}_tV$ at times 4.0 and 4.5.

Calculate your approximation of ${}_4V$ using this methodology.

(A) 71.0 (B) 71.4 (C) 71.9 (D) 72.4 (E) 72.8

2.11. For a 5-year warranty on Kira's new cell phone, you are given:

(i) The warranty pays 100 at the moment of breakage, if the phone breaks. The warranty only pays for one breakage.
(ii) If the phone has not broken, the warranty pays 100 at the end of 5 years.
(iii) Premiums of G are payable continuously at an annual rate of 25 until the phone breaks.
(iv) The force of breakage for this phone is $\mu_t = 0.02t$, $t \geq 0$.
(v) $\delta = 0.05$
(vi) ${}_tV$ denotes the gross premium policy value at time t for this warranty.
(vii) At the end of year 4, Kira's cell phone has not broken.
(viii) You approximate ${}_4V$ using Euler's method, with step size $h = 0.5$ and using the derivatives of ${}_tV$ at times 4.5 and 5.0.

Calculate your approximation of ${}_4V$ using this methodology.

(A) 71.0 (B) 71.4 (C) 71.9 (D) 72.4 (E) 72.8

Exercises continue on the next page . . .

2.12. **[150-S90:A2]** For a fully discrete life insurance of 1 on (x), ${}_{k+t}V$ is the policy value at duration $k+t$, where k is an integer and $0<t<1$, and b_{k+1} is the benefit payable at time $k+1$ for death in year $k+1$. Deaths are uniformly distributed over each year of age.

(a) (1 point) Use the survival function, $S(x)$, to demonstrate that

$$ {}_{1-t}q_{x+k+t} = \frac{(1-t)q_{x+k}}{1-tq_{x+k}} $$

(b) (4 points) Use the prospective policy value formula

$$ {}_{k+t}V = (b_{k+1})(v^{1-t})({}_{1-t}q_{x+k+t}) + ({}_{k+1}V)(v^{1-t})({}_{1-t}p_{x+k+t}) $$

to demonstrate that

$$ {}_{k+t}V = \frac{v^{1-t}}{1-tq_{x+k}}\Big((1-t)({}_kV+\pi_k)(1+i)+(t)({}_{k+1}V)(p_{x+k})\Big) $$

Additional old SOA Exam MLC questions: S12:21, F12:6, S17:10, S18:15

Additional old SOA Exam LTAM questions: F20:20, F21-B:15

Solutions

2.1. (a) $Z=v^T$.

(b) For $\mathbf{E}[Z \mid T\le h]$, the numerator of the conditional expectation is v^t times the conditional density function, which here is ${}_tp_x\,\mu_{x+t}/{}_hq_x$ if $t\le h$, 0 otherwise. So we get the parenthesized integral for $\mathbf{E}[Z\mid T\le h]$. Then $\Pr(T\le h)={}_hq_x$ and $\Pr(T>h)={}_hp_x$. The conditional density function for $Z\mid T>h$ is ${}_tp_x\,\mu_{x+t}/{}_hp_x$ for $t>h$, 0 otherwise. So

$$ \mathbf{E}[Z\mid T>h]=\int_h^\infty v^t\,\frac{{}_tp_x}{{}_hp_x}\mu_{x+t}\,\mathrm{d}t $$

But ${}_tp_x/{}_hp_x={}_{t-h}p_{x+h}$, so the integral becomes

$$ \int_h^\infty v^t\;{}_{t-h}p_{x+t}\,\mu_{x+t}\mathrm{d}t=\int_0^\infty v^{h+t'}\;{}_{t'}p_{x+h}\,\mu_{x+h+t'}\,\mathrm{d}t' $$

where we changed the variable $t'=t-h$. Factor out v^h and the rest of the integral is $\bar{A}_{x+h}$. This gets us the expression for $\bar{A}_x$ given in the question.

(c) From (b), the numerator of the limit expression is

$$ \bar{A}_{x+h}-\bar{A}_x=\bar{A}_{x+h}(1-v^h\,{}_hp_x)-\int_0^h v^t\,{}_tp_x\,\mu_{x+t}\,\mathrm{d}t $$

Divide through by h and calculate the limit of each of two terms. The limit of the first term is the product of the limit of $\bar{A}_{x+h}$, which is $\bar{A}_x$, and the limit of $(1-v^h\,{}_hp_x)/h$.

$$ \begin{aligned} \lim_{h\to0}\frac{1-v^h\,{}_hp_x}{h} &= \lim_{h\to0}\frac{v^0\,{}_0p_x-v^h\,{}_hp_x}{h} \\ &= -\frac{\mathrm{d}}{\mathrm{d}h}v^h\,{}_hp_x \quad \text{evaluated at } h=0 \\ &= -\frac{\mathrm{d}}{\mathrm{d}h}e^{-h(\delta+\int_0^h\mu_{x+u}\mathrm{d}u)} \quad \text{evaluated at } h=0 \end{aligned} $$

$$= \delta + \mu_x$$

For the second term, the derivative of the integral is the function evaluated at h, or $v^h\,{}_hp_x\,\mu_{x+h}$, which goes to μ_x. So the limit is $\bar{A}_x(\delta + \mu_x) - \mu_x$.

This is a special case of Thiele's equation for a single-premium policy.

2.2. By Thiele's equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{10}V = 0.04(122) + (20 - 0.06(20) - 0.4) - (1000 + 15 - 122)(0.005) = \boxed{\mathbf{18.815}}$$

2.3. By the form of Thiele's equation for net premium policy values,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}_5V = \delta\,{}_5V + P_5 - (b_5 - {}_5V)\mu_{60}$$

Because the policy is in the deferral period, $b_5 = 0$.

$$6240 = 4800 + 32{,}000\delta + 0.0032(32{,}000)$$
$$\delta = \frac{6240 - 0.0032(32{,}000) - 4800}{32{,}000} = \boxed{\mathbf{0.0418}}$$

2.4. By Thiele's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{15}V = \delta\,{}_{15}V + P_{15} - (b_{15} - {}_{15}V)\mu_{40}$$
$$-4.80 = 0.06\,{}_{15}V + 700 - (100{,}000 - {}_{15}V)(0.008)$$
$${}_{15}V = \frac{-4.80 - 700 + 800}{0.06 + 0.008} = \boxed{\mathbf{1400}}$$

2.5. Use equation (2.3) twice.

$$\mu_{64.9} = 0.002(1.01^{64.9}) = 0.003815$$
$${}_{19.9}V = \frac{1000 - 0.1\big(40 - 2 - 1000(0.003815)\big)}{1 + 0.1(0.003815 + 0.05)} = 991.2471$$
$$\mu_{64.8} = 0.002(1.01^{64.8}) = 0.003811$$
$${}_{19.8}V = \frac{991.2471 - 0.1\big(40 - 2 - 1000(0.003811)\big)}{1 + 0.1(0.003811 + 0.05)} = \boxed{\mathbf{982.54}}$$

2.6. Use equation (2.3) twice.

$$\mu_{64.9} = 0.001(1.015^{64.9}) = 0.002628$$
$${}_{9.9}V = \frac{-0.1\big(250 - 7.5 - 100{,}100(0.002628)\big)}{1 + 0.1(0.04 + 0.002628)} = 2.048805$$
$$\mu_{64.8} = 0.001(1.015^{64.8}) = 0.002624$$
$${}_{9.8}V = \frac{2.048805 - 0.1\big(250 - 7.5 - 100{,}100(0.002624)\big)}{1 + 0.1(0.04 + 0.002624)} = \boxed{\mathbf{4.0499}}$$

Practice Exam 1

1. (10 points) A disability insurance is modeled with a 4-state Markov chain having the following states:

0: Healthy
1: Disabled
2: Surrendered
3: Dead

Nonzero forces of transition for $x < 150$, where x is age, are

$$\mu_x^{01} = \frac{0.5}{150 - x}$$
$$\mu_x^{02} = \frac{0.4}{150 - x}$$
$$\mu_x^{03} = \frac{0.1}{150 - x}$$
$$\mu_x^{13} = 0.05$$

As a result of these forces of transition,

$$_tp_x^{00} = \frac{150 - x - t}{150 - x} \qquad 0 \le x < 150$$

(a) (2 points) Calculate the probability that a person healthy at age 50 will become disabled within 10 years.

(b) (2 points) Calculate the probability that a person healthy at age 50 will be disabled at age 60.

(c) (3 points) A disability insurance pays 40,000 per year continuously while an individual is disabled until the individual's 70$^{\text{th}}$ birthday.

Calculate the actuarial present value at $\delta = 0.05$ of a policy sold to a healthy individual age 50.

(d) (3 points) A disability insurance pays 40,000 per year continuously while an individual is disabled until the individual's 70$^{\text{th}}$ birthday. Benefits are subject to a 6 month waiting period.

Calculate the actuarial present value at $\delta = 0.05$ of a policy sold to a healthy individual age 50.

Exam questions continue on the next page . . .

2. (8 points) An individual age x is subject to a serious disease. The individual's status is modeled using these three states;

0: Healthy
1: Diseased
2: Dead

Forces of transition between states are

$$\mu_t^{01} = 0.02 \qquad \mu_t^{02} = 0.01 \qquad \mu_t^{10} = 0.20 \qquad \mu_t^{12} = 0.35$$

The effective annual rate of interest is 0.05.

Use Euler's method with step 0.5 to solve Kolmogorov's forward equations for the state probabilities for the first two years.

(a) (2 points) Using approximate probabilities, calculate $\ddot{a}^{(2)00}_{x:\overline{2}|}$.

(b) (2 points) Using Woolhouse's formula to two terms, calculate $\ddot{a}^{(12)00}_{x:\overline{2}|}$.

Forces of transition for the first two years are estimated assuming constant forces of transition within each year, and using the following data:

- 92 lives in state 0 remained in state 0 for 2 years.
- 2 lives in state 0 died, one at time 0.8 and one at time 1.5.
- 3 lives in state 0 moved to state 1 at times 0.8, 1.2, 1.6. They all stayed there to time 2.
- 8 lives in state 1 remained in state 1 for 2 years.
- 2 lives in state 1 moved to state 0, one at time 0.4 and one at time 1.8. They all stayed there to time 2.
- 4 lives in state 1 died at times 0.3, 0.6, 1.2, 1.7

(c) (2 points)

(i) Estimate μ_x^{01} and μ_{x+1}^{01}.

(ii) Estimate the standard deviations of the estimates of μ_x^{01} and μ_{x+1}^{01}.

(d) (2 points) Using estimates of the forces of transition, calculate the probability of staying healthy for 2 years.

3. (7 points) A universal life policy on (40) pays a death benefit of 10,000 plus the account value *at the end of the previous year*. If death occurs in the first year of the policy, it pays 10,000.

You are given:

(i) Expense charges are 30% of premium plus 50 in the first year, 5% of premium plus 10 in renewal years.

(ii) COI rate is $q_{40+t} = 0.006 + 0.001t$.

(iii) Interest rate for COI computations is 0.03.

(iv) Interest is credited at $i = 0.04$.

The policyholder pays 200 at the start of each year.

(a) (3 points) Calculate the account values at the ends of the first three years.

(b) (3 points) Calculate the account values at the ends of the first three years for an otherwise similar policy that pays a death benefit of 10,000 plus the account value at the end of the current year.

(c) (1 point) Explain why the account values in (b) are less than the corresponding account values in (a).

Exam questions continue on the next page . . .

4. (8 points) For two independent lives:

(i) The first life is subject to a force of accidental death of 0.01 and a force of death due to other causes of 0.03.
(ii) The second life is subject to a force of accidental death of 0.005 and a force of death due to other causes of 0.02.
(iii) $\delta = 0.04$

(a) (2 points) Calculate the actuarial present value of an insurance paying 1000 upon the first death if the death is by accident.

(b) (2 points) Calculate the actuarial present value of an insurance paying 1000 upon the second death if the death is by accident.

(c) (2 points) Calculate the actuarial present value of an insurance paying 1000 upon the second death if both deaths were by accident.

(d) (2 points) Calculate the actuarial present value of an insurance paying 1000 upon the second death if the second life dies second.

5. (6 points) A profit test is performed on a life insurance policy on (50) and (48). The policy pays 10,000 at the end of the year of the death of the last survivor of the two lives.

For the fifth year of the policy, assuming both lives are alive at the beginning of the year, you are given the following profit test assumptions:

(i) $q_{52} = 0.006$; $q_{54} = 0.01$; $q_{52:54} = 0.012$
(ii) Premium is 150.
(iii) Expenses are 10.
(iv) Reserve at the beginning of the year is 620.
(v) Reserve at the end of the year is 750 if both are alive, 2012 if only (48) is alive, 2228 if only (50) is alive.
(vi) There are no surrenders.
(vii) $i = 0.06$

For 1000 such policies with both lives alive at the beginning of the fifth year, the following experience occurs:

(i) For five policies, only (48) dies. For another seven policies, only (50) dies. There are no policies for which both die.
(ii) Expenses are 8.
(iii) Interest earned is at a rate of 0.055.

(a) (2 points) Calculate the anticipated profit in the fifth year per policy in force at the beginning of the year, given that both lives are alive at the beginning of the year.

(b) (2 points) Calculate the fifth year gain per policy in force at the beginning of the year, given that both lives are alive at the beginning of the year.

(c) (2 points) Assuming that both lives are alive at the beginning of the year, the reserve at the beginning of the fifth year is changed so that anticipated profit is 0.

Determine the revised reserve.

Exam questions continue on the next page . . .

6. (9 points) An equity-linked insurance is issued to a life age 60. It matures at age 65. It has the following features:

(i) No front-end load.
(ii) A management charge of 2% of the fund value, deducted at the beginning of each year.
(iii) A surrender charge of 4%, 3%, 2%, 1% of the fund value in years 1, 2, 3, 4 respectively.

Upon death, the surrender charge is waived. The death benefit is paid at the end of the year.

You are given:

(i) Mortality rate is $q_{60+t} = 0.006 + 0.001t$, $t = 0, 1, 2, 3, 4$.
(ii) Independent surrender rate is 0.03 each year for the first 4 years.
(iii) Surrenders occur at the end of the year.

A single premium of 100,000 is paid.

(a) (1 point) The value of the waiver of the surrender charge may be evaluated as a decreasing insurance at a certain interest rate.
Determine the interest rate.

(b) (2 points) Calculate the value at issue of the waiver of the surrender charge.

For part (c) only, use the following information:

The company adds a GMMB to the policy. 110% of the fund is paid at maturity.

(c) (2 points) Calculate the value at issue of the GMMB.

For parts (d)–(e) only, use the following information:

The company enhances the GMDB. 120% of the fund is paid at the end of the year of death.

(d) (2 points) Calculate the value at issue of the enhanced GMDB, including the value of the waiver of the surrender charge.

(e) (2 points) Calculate the portion of the management charge, as a level percentage of the fund value, needed to fund the GMDB.

Exam questions continue on the next page . . .

7. (11 points) In a defined contribution plan, the employer and employee each contribute 3% of salary at the end of each year. You are given:

(i) Salaries increase 4% each year.
(ii) The pension fund earns 5% effective interest each year.

Vera enters the plan at age 30 at 70,000 salary. At retirement at age 65, Vera will convert the fund into a monthly whole life annuity-due priced with the following assumptions:

- Mortality follows the Standard Ultimate Life Table.
- $i = 0.05$
- Woolhouse's formula to two terms is used for fractional payments.

(a) (3 points) Calculate the replacement ratio.

(b) (3 points) For this part only, assume that Vera's salary increases 3% each year for the first 5 years, and 4% each year thereafter.
Calculate the replacement ratio if retirement occurs at age 65.

(c) (3 points) For this part only, assume that the pension fund earns 6% effective interest in all years.
Calculate the replacement ratio if retirement occurs at age 65.

(d) (2 points) Vera dies at age 55. Her husband, age 50, inherits the fund and purchases a monthly whole-life annuity-due. The assumptions used to calculate the single premium for this annuity are the same as the assumptions used by Vera.
Calculate the monthly benefit paid by the annuity.

Solutions to the above questions begin on page 731.

Appendix A. Solutions to the Practice Exams

Practice Exam 1

1. (a) **[Section 4.1]**

$$\int_0^{10} {}_tp_{50}^{00}\,\mu_{50+t}^{01}\,dt = \int_0^{10}\left(\frac{100-t}{100}\right)\left(\frac{0.5}{100-t}\right)dt$$
$$= \int_0^{10}\frac{0.5\,dt}{100} = \boxed{\mathbf{0.05}}$$

(b) **[Section 4.1]**

$$\int_0^{10} {}_tp_{50}^{00}\,\mu_{t0+t}^{01}\,{}_{10-t}p_{50+t}^{11}\,dt = \int_0^{10}\left(\frac{100-t}{100}\right)\left(\frac{0.5}{100-t}\right)e^{-0.05(10-t)}\,dt$$
$$= 0.005e^{-0.5}\int_0^{10} e^{0.05t}dt$$
$$= \frac{0.005e^{-0.5}}{0.05}(e^{0.5}-1) = \boxed{\mathbf{0.039347}}$$

(c) **[Section 5.1]** The probability of state 1 at time t is

$${}_tp_{50}^{01} = \int_0^t\left(\frac{100-u}{100}\right)\left(\frac{0.5}{100-u}\right)e^{-0.05(t-u)}\,du$$
$$= 0.005e^{-0.05t}\int_0^t e^{0.05u}\,du$$
$$= 0.005e^{-0.05t}\left(\frac{e^{0.05t}-1}{0.05}\right)$$
$$= 0.1(1-e^{-0.05t})$$

Thus the EPV of the disability annuity is

$$\int_0^{20} v^t\,{}_tp_0^{01}\,dt = 0.1\int_0^{20} e^{-0.05t}(1-e^{-0.05t})dt$$
$$= 0.1\left(\frac{1-e^{-1}}{0.05}-\frac{1-e^{-2}}{0.1}\right) = 0.399576$$

The actuarial present value of the disability insurance is 40,000(0.399576) = $\boxed{\mathbf{15{,}983}}$

(d) **[Section 6.1]** We must use continuous sojourn annuities to account for the waiting period. Since μ^{13} is constant and equal to 0.05 and δ is also 0.05, the EPV of a 6-month deferred annuity at time t to time 20 (age 70) in that state is

$${}_{0.5|}\bar{a}^{\overline{11}}_{50+t:\overline{19.5-t|}} = e^{-0.5(0.05+0.05)}\frac{1-e^{-0.1(19.5-t)}}{0.05+0.05} = 10e^{-0.05}(1-e^{-0.1(19.5-t)})$$

Also

$${}_tp_{50}^{00}\,\mu_{50+t}^{01} = \frac{100-t}{100}\left(\frac{0.5}{100-t}\right) = \frac{1}{200}$$

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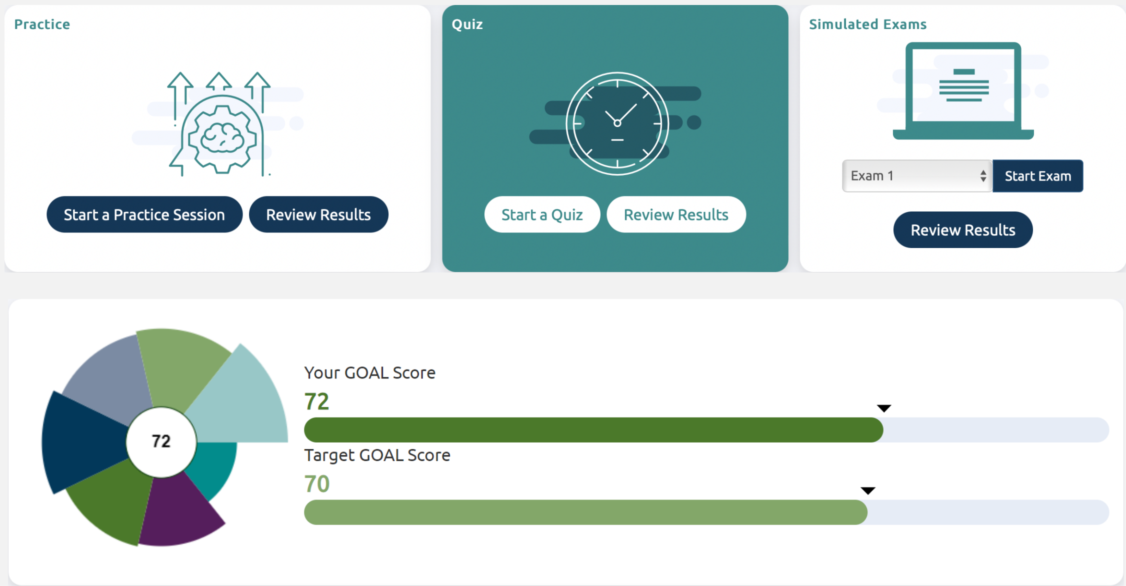

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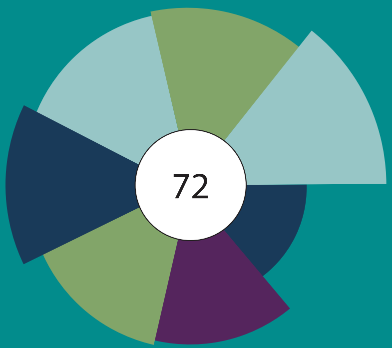

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