Section B: Risk Measures

Value-at-Risk, Jorion

One thing to always keep in mind when reading this text is that it is focused on the banking industry. It mainly focuses on market and credit risk. It also has a very short time horizon. For example the author calculates a 10-day VaR. On exam day it is important to be flexible. If you are given a scenario where Life Insurance Company ABC is in the context, you'll want to recommend a risk metric with a much longer time horizon.

Chapter 5: Computing VaR

Key Concepts

- Understand the three ways to calculate VAR
- Be able to calculate estimation errors
- Understand pros and cons of EVT

Introduction

- VaR is the maximum amount that will be lost over a certain period with a certain degree of confidence
- Measures downside risk
- Three ways to calculate VaR:
    - Non-parametric
    - Parametric
    - EVT (semi-parametric)

1) Computing VaR
   a) VaR is the worst loss over a target horizon such that there is a low (and specified) probability that the actual loss is greater than the "worst loss"
      i) If c is confidence level and L is loss → $P(L>VaR) \leq 1 - c$

   b) Steps in computing VaR (looking at 100M portfolio over 10 days at 99% CI)
      i) Mark to market the current portfolio (100M)
      ii) Measure variability of the risk factor (e.g. 15% per year)
      iii) Set the time horizon (e.g. adjust to 10 days)
      iv) Set confidence level (e.g. 99%, which yields 2.33 factor, assuming normal)
      v) Report worst potential loss (e.g. 7M @ 99CI)
         (1) $100M * 15\% * (10/252)^{1/2} * 2.33 = 7M$
         (2) Note that with IID returns, variances are additive, implying that volatility grows with the square root of time (using trading days instead of calendar days)

   c) Nonparametric VaR
      i) Makes no assumption regarding shape of distribution of returns
      ii) W_0 is initial investment and R is return
      iii) VaR expresses worst loss at some CI, but relative to what? (mean or zero?)
      iv) Relative (compared to E(W)) and absolute VAR (compared to 0) are given below

$$\text{VaR(mean)} = E(W) - W^* = -W_0(R^* - \mu) \quad (5.2)$$

$$\text{VaR(zero)} = W_0 - W^* = -W_0 R^* \quad (5.3)$$

There is not a right or wrong answer. It really depends on the context. A lot of times you'll see VaR computed as the Xth percentile less the mean.

v) Relative VAR is more appropriate as it views risk in terms of a deviation from "budget"

$$\text{C} = \int_{W^*}^{\infty} f\,(w)dw \quad (5.4)$$

$$1 - \text{C} = \int_{-\infty}^{W^*} f\,(w)dw = P(w \leq W^*) = P \quad (5.5)$$

vi) W^* is the quantile of the distribution (aka percentile)

Text walks through an example of determining a VAR without parameterization.

d) Parametric VaR
   i) VaR can be simplified if we assume a distribution that has parameters
   ii) If we pick a normal distribution, we need to:
      (1) Translate f(w) into standard normal
      (2) Associate W* with cutoff return R* such that $W^*=W_0\,(1+R^*)$

$$-\alpha = \frac{-|R| - \mu}{\sigma} \quad (5.6)$$

The above formula is just the classic "Z = (X – μ) / σ" formula. In this text the author uses alpha for Z. You'll see a lot of 1.645, 1.96, and 2.33, which are just different Z values on the standard normal distribution.

$$1 - \text{c} = \int_{-\infty}^{W^*} f\,(w)dw = \int_{-\infty}^{-|R^*|} f\,(r)dr = \int_{-\infty}^{-\alpha} \phi\,(\epsilon)d\epsilon \quad (5.7)$$

$$p = \text{N(x)} = \int_{-\infty}^{x} \phi\,(\epsilon)d\epsilon \quad (5.8)$$

      (3) With the deviate, α, we can calculate the cutoff return, R*

$$R^* = -\alpha\sigma = \mu \quad (5.9)$$

      (4) Using (5.2), we can calculate the VaR relative to the mean

$$\text{VaR(mean)} = -W_0(R^* - \mu) = W_0 \alpha\sigma\sqrt{\Delta t} \quad (5.10)$$

   iii) Text shows that, in this case, sample JP Morgan data yields similar VaR for both parametric and non-parametric version (normal often does a good job for CI<99%)
   iv) Using another distribution (e.g. student t) requires a different multiplier for α

e) Why VaR as a risk measure
  i) Can express risk in one number
  ii) Flexible – can choose various time horizons and/or levels of confidence
  iii) Very popular measure of risk
  iv) Properties of coherent risk measures:
    (1) Monotonicity à if $W1 \leq W2$, then $p(W1) \leq p(W2)$
    (2) Translation invariance à $p(W+k) = p(W) - k$
    (3) Homogeneity à $p(IW) = b p(W)$
    (4) Subadditivity à $p(W1+W2) \leq p(W1) + p(W2)$
  v) VaR fails Subadditivity
  vi) ETL (CTE) satisfies all 4 of these
    (1) ETL is likely to be an additional measure used in many circumstances
  vii) Disadvantages of VaR
    (1) Does not describe the shape of the tail
    (2) Not coherent

2) Choice of quantitative factors
  a) Longer time horizon or higher level of confidence à Higher VaR
  b) VaR as a benchmark measure à VaR as a yardstick is arbitrary, banking has settled on 99CI and daily horizon
  c) VaR as a potential loss measure
    i) Liquidation period à daily horizon makes sense since people turn over their portfolios frequently
    ii) VaR assumes portfolio is static over the time horizon, so daily makes sense
    iii) Daily VaR is comparable to daily income statement numbers

  d) VaR as equity capital
    i) Very important as loss above VaR would wipe out capital, leading to bankruptcy
    ii) Must assume VaR captures all risks à this is a stretch
    iii) CI chosen based on risk aversion and cost of exceeding VaR
    iv) Time horizon should reflect ability to take corrective action (e.g. raise capital)
    v) VaR ignores interim losses that might exceed quantile (causing liquidation)
      (1) Could introduce a MaxVAR higher than VaR such that interim levels are not allowed to breach MaxVAR
      (2) Table 5-2 shows MaxVaR for normal distribution

  e) Criteria for backtesting
    i) Comparing VaR with subsequent income statement allows can help identify biases
    ii) Shorter horizons increase the power of the tests (more observations available)
    iii) Too high of a CI reduces number of observations in the tail (95% should suffice for backtesting)

  f) Application: The Basel Parameters
    i) Basel chose 99CI over 10 day horizon à balances enough time to detect problems with costs of frequent monitory AND balances sound financial with cost to banks
    ii) The capital amount is adjusted by a factor, k, of 3
      (1) This factor is arbitrary and serves similar purpose to higher CI

g) Conversion of VaR parameters
   i) Using normal is convenient as it allows conversion to various CIs (via α)
   ii) If VaR is calculated at the 95% confidence level and you want the 99% confidence level, you just need to scale by (α_{99} / α_{95})

3) Assessing VaR Precision
   a) The problem of measurement errors
      i) It is useful to know how confident we are in the VaR measure
         (1) VaR is only an estimate of the true VaR of the underlying (and unknown) distribution
         (2) If our VaR estimate is 15M, how sure are we that the actual VaR is within (13M, 17M)?

   b) Estimation errors in means and variances
      i) If underlying distribution is normal, then the distribution of sample mean and variance is known

$$\hat{\mu} \sim N(\mu, \sigma^2/T) \quad (5.12)$$

$$\frac{(T-1)\hat{\sigma}^2}{\sigma^2} \sim x^2(T-1) \quad (5.13)$$

      ii) As T becomes large (above 20), chi squared converges quickly to a normal

$$\hat{\sigma}^2 \sim N(\sigma^2, \sigma^4 \frac{2}{T-1}) \quad (5.14)$$

      iii) Sample std deviation in large samples is:

$$\mathrm{SE}(\hat{\sigma}) = \sigma \sqrt{\frac{1}{2T}} \quad (5.15)$$

      iv) If sample mean/variance is -0.15% and 3.39% with T=384, one standard error = 3.39 * $(1/384)^{1/2}$ = 0.17%, so sample mean is imprecise
      v) Conversely, SE for σ = 3.39 * $(1/768)^{1/2}$ = 0.12%, giving a solid estimate of σ
      vi) As observations increase, the sample should approach true value
      vii) Confidence bands for sigma based quantile can be described by

$$\hat{q}_\sigma = \alpha \hat{\sigma} \quad (5.16)$$

   c) Estimation error in sample quantiles
      i) For arbitrary distributions, Kendall gives us a nonparametric approach

$$\mathrm{SE}(\hat{q}) = \sqrt{\frac{c(1-c)}{Tf(q)^2}} \quad (5.17)$$

      ii) As T increases, same quantiles become more reliable
      iii) Sample quantiles become less and less reliable as you approach the tail
      iv) Formula (5.17) has little use when the distribution is unknown
         (1) However, SE can be measured using bootstrapping

(2) Bootstrapping involves resampling T observations K times
(3) Using these K statistics, we can get a distribution to help understand how much variation exists in the sample statistic
(4) This approach shows that estimation error is much larger with ETL than VaR

d) Comparison of methods
   i) We have developed two approaches for measuring VaR
      (1) Reading quantile directly from distribution q and
      (2) Calculating the standard deviation and then scaling by $\alpha\sigma$
   ii) Which way is best?
      (1) There is substantial estimation error in estimated quantiles, particularly w/ high CI
      (2) Parametric methods are inherently more precise because σ reflects each sample
      (3) It is difficult to know which distribution to choose if using parametric

4) Extreme-value theory (semi-parametric approach à only works for the tails)
   a) The EVT distribution
      i) EVT theorem à Generalized Pareto
         (1) Need to pick a cutoff point (u)
         (2) Cumulative distribution function:

$$\mathrm{F(y)} = 1 - (1 + \xi \mathrm{y})^{-\frac{1}{\xi}} \quad \xi \neq 0$$

$$F(y) = 1 - \exp(-y) \quad \xi = 0 \quad (5.18)$$

      ii) Where $y = (x-\mu) / \beta$ with $\beta>0$ (*with this distribution you are modeling the excess over u*)
      iii) ξ determines how fast the tail will disappear
      iv) $\xi=0$ corresponds to a normal distribution
      v) $\xi>0$ implies fatter tails (0.2 / 0.4 apply for stock market)
   b) Quantiles and ETL
      i) The tail distribution and density functions are

$$F(x) = 1 - \left(\frac{N_u}{N}\right)\left[1 + \frac{\xi}{\beta}(x-u)\right]^{-1/\xi} \quad (5.19)$$

$$f(x) = \left(\frac{N_u}{N}\right)\left(\frac{1}{\beta}\right)\left[1 + \frac{\xi}{\beta}(x-u)\right]^{-(1/\xi)-1} \quad (5.20)$$

      ii) Which yields the following quantile estimator of VaR and sample ETL

$$\widehat{\mathrm{VaR}} = u + \frac{\hat{\beta}}{\hat{\xi}}\left\{[(N/N_u)(1-c)]^{-\hat{\xi}} - 1\right\} \quad (5.21)$$

$$\widehat{\mathrm{ETL}} = \frac{\widehat{\mathrm{VaR}}}{1-\hat{\xi}} + \frac{\hat{\beta} - \hat{\xi} u}{1-\hat{\xi}} \quad (5.22)$$

As you can see, if the distribution is not normal these formulas get pretty nasty. This is why throughout the syllabus you'll see VaR and CTE being calculated via simulation method.

iii) Empirical distribution just reflects history, so quantiles are very imprecisely estimated
iv) Further, normal distribution fit assigns too low of a probability to the tail
v) EVT provides a smooth parametric fit to the data, without imposing unnecessary or unrealistic assumptions

c) Time aggregation
   i) If returns are normal, we know we can adjust with scalar = $T^{1/2}$
   ii) Scalar increases at roughly T^{ξ}
   iii) Extreme values aggregate at a slower rate than normal distribution as horizon increases, making Basel square-root-of-time approach excessive
   iv) EVT is univariate, so it can't help characterize joint distributions

d) EVT evaluation
   i) Useful for estimating tail probabilities of extreme events
   ii) Empirical data doesn't have enough tail samples to estimate VaR reliably
   iii) EVT helps us smooth out the tail
   iv) Need to use stress testing with EVT

Appendix
1) Justification for Basel multiplier
2) This section works through some algebra that supports the Basel multiplier of k=3

Text Review Questions

You should use review questions at the end of chapters to gauge how well you absorbed the material. The answers to the review questions are published – if you don't have them Google "jorion value at risk end of chapter" and it will be one of the first hits.

Another good idea is to come back to these review questions a few days after reading the material and see how much you remember. Repetition is one of the keys to transferring information into your long term memory.

Most of the questions are straight forward and therefore, do not need any comments.

1. There are 252 trading days in a year. To get the 1-day volatility, divide 25% by the square root of 252.

In order for problem #1 to be intuitive, I had to relate it back to the standard normal distribution that I learned back in stat 101.

Mean = 10M
Stdev = 25% * 10M / $\sqrt{252}$ = 0.1575

$$Z = \frac{X - \mu}{\sigma}$$

Back solving for X à 2.326 * 0.1575 + 10 à gives an answer of 10.365
So the VaR is the excess over the mean, or 0.365

4. Student T has a fatter tail than the normal, so it makes sense that α is larger. Keep in mind that for other distributions the formulas are not so nice and neat.