Contents

Lesson 7

Survival Distributions: Fractional Ages

Reading: *Actuarial Mathematics for Life Contingent Risks* 3.2 or *Models for Quantifying Risk* (4th edition) 6.6.1, 6.6.2

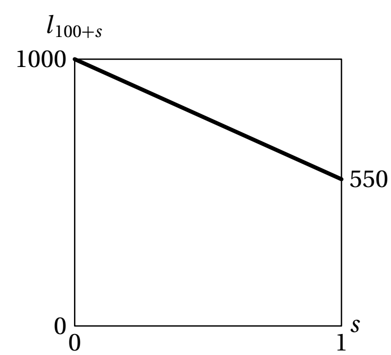
Life tables list mortality rates (q_x) or lives (l_x) for integral ages only. Often, it is necessary to determine lives at fractional ages (like $l_{x+0.5}$ for x an integer) or mortality rates for fractions of a year. We need some way to interpolate between ages.

7.1 Uniform distribution of deaths

The easiest interpolation method is linear interpolation, or uniform distribution of deaths between integral ages (UDD). This means that the number of lives at age $x+s$, $0 \le s \le 1$, is a weighted average of the number of lives at age x and the number of lives at age $x+1$:

$$l_{x+s} = (1-s)l_x + s\,l_{x+1} = l_x - s\,d_x \tag{7.1}$$

The graph of l_{x+s} is a straight line between $s=0$ and $s=1$ with slope $-d_x$. The graph at the right portrays this for a mortality rate $q_{100} = 0.45$ and $l_{100} = 1000$.



Contrast UDD with an assumption of a uniform survival function. If age at death is uniformly distributed, then l_x as a function of x is a straight line. If UDD is assumed, l_x is a straight line between integral ages, but the slope may vary for different ages. Thus if age at death is uniformly distributed, UDD holds at all ages, but not conversely.

Using l_{x+s}, we can compute ${}_sq_x$:

$$\begin{aligned} {}_sq_x &= 1 - {}_sp_x \\ &= 1 - \frac{l_{x+s}}{l_x} = 1 - (1 - sq_x) = sq_x \end{aligned} \tag{7.2}$$

That is one of the most important formulas, so let's state it again:

$$\boxed{{}_sq_x = sq_x} \tag{7.2}$$

More generally, for $0 \le s+t \le 1$,

$$\begin{aligned} {}_sq_{x+t} &= 1 - {}_sp_{x+t} = 1 - \frac{l_{x+s+t}}{l_{x+t}} \\ &= 1 - \frac{l_x - (s+t)d_x}{l_x - t\,d_x} = \frac{s\,d_x}{l_x - t\,d_x} = \frac{sq_x}{1-tq_x} \end{aligned} \tag{7.3}$$

where the last equation was obtained by dividing numerator and denominator by l_x. The important point to pick up is that while ${}_sq_x$ is the proportion of the year s times q_x, the corresponding concept at age $x+t$, ${}_sq_{x+t}$, is *not* sq_x, but is in fact higher than sq_x. The *number* of lives dying in any amount of time is constant, and since there are fewer and fewer lives as the year progresses, the *rate* of death is in fact increasing over the year. The numerator of ${}_sq_{x+t}$ is the proportion of the year being measured s times the death rate, but then this must be divided by 1 minus the proportion of the year that elapsed before the start of measurement.

For most problems involving death probabilities, it will suffice if you remember that l_{x+s} is linearly interpolated. It often helps to create a life table with an arbitrary radix. Try working out the following example before looking at the answer.

EXAMPLE 7A You are given:

(i) $q_x = 0.1$

(ii) Uniform distribution of deaths between integral ages is assumed.

Calculate ${}_{1/2}q_{x+1/4}$.

ANSWER: Let $l_x = 1$. Then $l_{x+1} = l_x(1-q_x) = 0.9$ and $d_x = 0.1$. Linearly interpolating,

$$l_{x+1/4} = l_x - \tfrac{1}{4}d_x = 1 - \tfrac{1}{4}(0.1) = 0.975$$
$$l_{x+3/4} = l_x - \tfrac{3}{4}d_x = 1 - \tfrac{3}{4}(0.1) = 0.925$$
$${}_{1/2}q_{x+1/4} = \frac{l_{x+1/4} - l_{x+3/4}}{l_{x+1/4}} = \frac{0.975 - 0.925}{0.975} = \boxed{\mathbf{0.051282}}$$

You could also use equation (7.3) to work this example. □

EXAMPLE 7B For two lives age (x) with independent future lifetimes, ${}_{k|}q_x = 0.1(k+1)$ for $k = 0, 1, 2$. Deaths are uniformly distributed between integral ages.

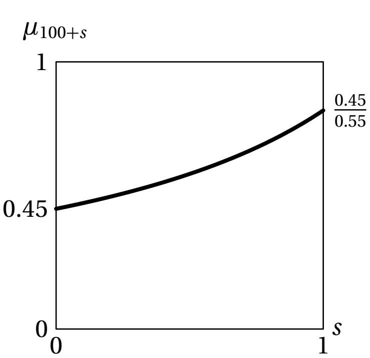
Calculate the probability that both lives will survive 2.25 years.

ANSWER: Since the two lives are independent, the probability of both surviving 2.25 years is the square of ${}_{2.25}p_x$, the probability of one surviving 2.25 years. If we let $l_x = 1$ and use $d_{x+k} = l_x\,{}_{k|}q_x$, we get

$$q_x = 0.1(1) = 0.1 \qquad l_{x+1} = 1 - d_x = 1 - 0.1 = 0.9$$
$${}_{1|}q_x = 0.1(2) = 0.2 \qquad l_{x+2} = 0.9 - d_{x+1} = 0.9 - 0.2 = 0.7$$
$${}_{2|}q_x = 0.1(3) = 0.3 \qquad l_{x+3} = 0.7 - d_{x+2} = 0.7 - 0.3 = 0.4$$

Then linearly interpolating between l_{x+2} and l_{x+3}, we get $l_{x+2.25} = 0.7 - 0.25(0.3) = 0.625$, and ${}_{2.25}p_x = l_{x+2.25}/l_x = 0.625$. Squaring, the answer is $0.625^2 = \boxed{\mathbf{0.390625}}$. □

The probability density function of T_x, ${}_sp_x\,\mu_{x+s}$, is the constant q_x, the derivative of the conditional cumulative distribution function ${}_sq_x = sq_x$ with respect to s. That is another important formula, since the density is needed to compute expected values, so let's repeat it:



$$\boxed{{}_sp_x\,\mu_{x+s} = q_x} \tag{7.4}$$

It follows that the force of mortality is q_x divided by $1 - sq_x$:

$$\mu_{x+s} = \frac{q_x}{{}_sp_x} = \frac{q_x}{1 - sq_x} \tag{7.5}$$

The force of mortality increases over the year, as illustrated in the graph for $q_{100} = 0.45$ to the right.

Quiz 7-1 You are given:

(i) $\mu_{50.4} = 0.01$
(ii) Deaths are uniformly distributed between integral ages.

Calculate ${}_{0.6}q_{50.4}$.

Complete Expectation of Life Under UDD

If the complete future lifetime random variable T is written as $T = K + S$, where K is the curtate future lifetime and S is the fraction of the last year lived, then K and S are independent. This is usually not true if uniform distribution of deaths is not assumed. Since S is uniform on $[0,1)$, $\mathbf{E}[S] = \frac{1}{2}$ and $\mathrm{Var}(S) = \frac{1}{12}$. It follows from $\mathbf{E}[S] = \frac{1}{2}$ that

$$\mathring{e}_x = e_x + \tfrac{1}{2} \tag{7.6}$$

More common on exams are questions asking you to evaluate the temporary complete expectancy of life under UDD. You can always evaluate the temporary complete expectancy, whether or not UDD is assumed, by integrating ${}_tp_x$, as indicated by formula (5.6) on page 80. For UDD, ${}_tp_x$ is linear between integral ages. Therefore, a rule we learned in Lesson 5 applies for all integral x:

$$\mathring{e}_{x:\overline{1}|} = p_x + 0.5q_x \tag{5.11}$$

This equation will be useful. In addition, the method for generating this equation can be used to work out questions involving temporary complete life expectancies for short periods. The following example illustrates this. This example will be reminiscent of calculating temporary complete life expectancy for uniform mortality.

EXAMPLE 7C You are given

(i) $q_x = 0.1$.
(ii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{x:\overline{0.4}|}$.

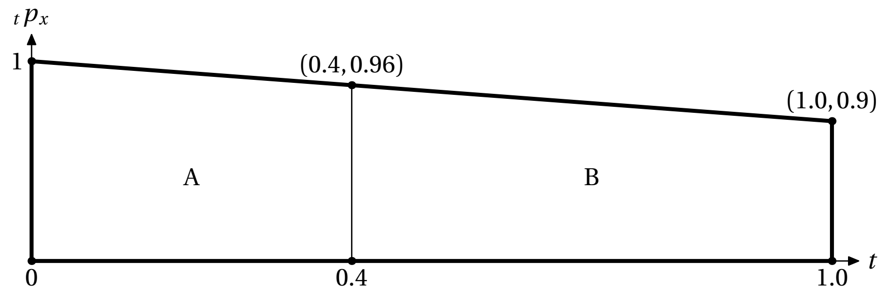
ANSWER: We will discuss two ways to solve this: an algebraic method and a geometric method.

The algebraic method is based on the double expectation theorem, equation (1.5). It uses the fact that *for a uniform distribution, the mean is the midpoint.* If deaths occur uniformly between integral ages, then those who die within a period contained within a year survive half the period on the average.

In this example, those who die within 0.4 survive an average of 0.2. Those who survive 0.4 survive an average of 0.4 of course. The temporary life expectancy is the weighted average of these two groups, or ${}_{0.4}q_x(0.2) + {}_{0.4}p_x(0.4)$. This is:

$$\begin{aligned}
{}_{0.4}q_x &= (0.4)(0.1) = 0.04 \\
{}_{0.4}p_x &= 1 - 0.04 = 0.96 \\
\mathring{e}_{x:\overline{0.4}|} &= 0.04(0.2) + 0.96(0.4) = \boxed{\mathbf{0.392}}
\end{aligned}$$

An equivalent geometric method, the trapezoidal rule, is to draw the ${}_tp_x$ function from 0 to 0.4. The integral of ${}_tp_x$ is the area under the line, which is the area of a trapezoid: the average of the heights times the width. The following is the graph (not drawn to scale):



Trapezoid A is the area we are interested in. Its area is $\frac{1}{2}(1+0.96)(0.4) = \boxed{\mathbf{0.392}}$. □

Quiz 7-2 As in Example 7C, you are given

(i) $q_x = 0.1$.
(ii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{x+0.4:\overline{0.6}|}$.

Let's now work out an example in which the duration crosses an integral boundary.

EXAMPLE 7D You are given:

(i) $q_x = 0.1$
(ii) $q_{x+1} = 0.2$
(iii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{x+0.5:\overline{1}|}$.

ANSWER: Let's start with the algebraic method. Since the mortality rate changes at $x+1$, we must split the group into those who die before $x+1$, those who die afterwards, and those who survive. Those who die before $x+1$ live 0.25 on the average since the period to $x+1$ is length 0.5. Those who die after $x+1$ live between 0.5 and 1 years; the midpoint of 0.5 and 1 is 0.75, so they live 0.75 years on the average. Those who survive live 1 year.

Now let's calculate the probabilities.

$$
\begin{aligned}
{}_{0.5}q_{x+0.5} &= \frac{0.5(0.1)}{1-0.5(0.1)} = \frac{5}{95} \\
{}_{0.5}p_{x+0.5} &= 1 - \frac{5}{95} = \frac{90}{95} \\
{}_{0.5|0.5}q_{x+0.5} &= \left(\frac{90}{95}\right)\big(0.5(0.2)\big) = \frac{9}{95} \\
{}_{1}p_{x+0.5} &= 1 - \frac{5}{95} - \frac{9}{95} = \frac{81}{95}
\end{aligned}
$$

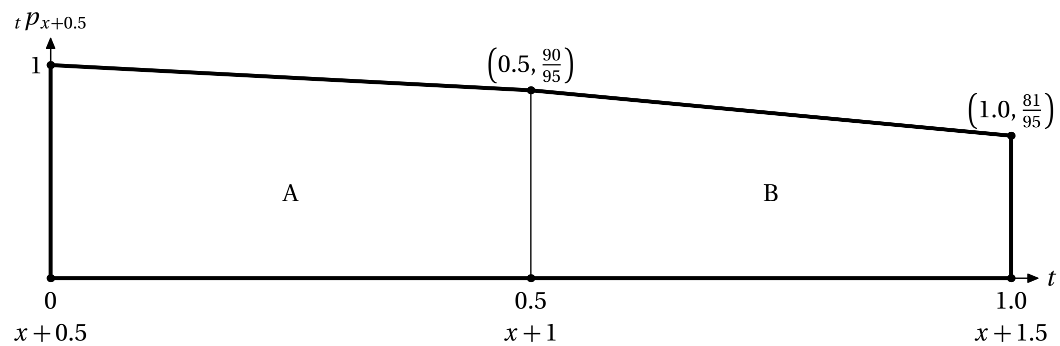
These probabilities could also be calculated by setting up an l_x table with radix 100 at age x and interpolating within it to get $l_{x+0.5}$ and $l_{x+1.5}$. Then

$$\begin{aligned}
l_{x+1} &= 0.9 l_x = 90\\
l_{x+2} &= 0.8 l_{x+1} = 72\\
l_{x+0.5} &= 0.5(90+100) = 95\\
l_{x+1.5} &= 0.5(72+90) = 81\\
{}_{0.5}q_{x+0.5} &= 1 - \frac{90}{95} = \frac{5}{95}\\
{}_{0.5|0.5}q_{x+0.5} &= \frac{90-81}{95} = \frac{9}{95}\\
{}_{1}p_{x+0.5} &= \frac{l_{x+1.5}}{l_{x+0.5}} = \frac{81}{95}
\end{aligned}$$

Either way, we're now ready to calculate $\mathring{e}_{x+0.5:\overline{1}|}$.

$$\mathring{e}_{x+0.5:\overline{1}|} = \frac{5(0.25)+9(0.75)+81(1)}{95} = \boxed{\mathbf{\frac{89}{95}}}$$

For the geometric method we draw the following graph:



The heights at $x+1$ and $x+1.5$ are as we computed above. Then we compute each area separately. The area of A is $\frac{1}{2}\left(1+\frac{90}{95}\right)(0.5) = \frac{185}{95(4)}$. The area of B is $\frac{1}{2}\left(\frac{90}{95}+\frac{81}{95}\right)(0.5) = \frac{171}{95(4)}$. Adding them up, we get $\frac{185+171}{95(4)} = \boxed{\mathbf{\frac{89}{95}}}$. □

Quiz 7-3 The probability that a battery fails by the end of the kth month is given in the following table:

k	Probability of battery failure by the end of month k
1	0.05
2	0.20
3	0.60

Between integral months, time of failure for the battery is uniformly distributed.
Calculate the expected amount of time the battery survives within 2.25 months.

To calculate $\mathring{e}_{x:\overline{n}|}$ in terms of $e_{x:\overline{n}|}$ when x and n are both integers, note that those who survive n years contribute the same to both. Those who die contribute an average of $\frac{1}{2}$ more to $\mathring{e}_{x:\overline{n}|}$ since they die on the average in the middle of the year. Thus the difference is $\frac{1}{2}\,{}_nq_x$:

$$\mathring{e}_{x:\overline{n}|} = e_{x:\overline{n}|} + 0.5\,{}_nq_x \tag{7.7}$$

EXAMPLE 7E You are given:

(i) $q_x = 0.01$ for $x = 50, 51, \ldots, 59$.

(ii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{50:\overline{10}|}$.

ANSWER: As we just said, $\mathring{e}_{50:\overline{10}|} = e_{50:\overline{10}|} + 0.5\,{}_{10}q_{50}$. The first summand, $e_{50:\overline{10}|}$, is the sum of ${}_kp_{50} = 0.99^k$ for $k = 1, \ldots, 10$. This sum is a geometric series:

$$e_{50:\overline{10}|} = \sum_{k=1}^{10} 0.99^k = \frac{0.99 - 0.99^{11}}{1 - 0.99} = 9.46617$$

The second summand, the probability of dying within 10 years is ${}_{10}q_{50} = 1 - 0.99^{10} = 0.095618$. Therefore

$$\mathring{e}_{50:\overline{10}|} = 9.46617 + 0.5(0.095618) = \boxed{\mathbf{9.51398}}$$

□

7.2 Constant force of mortality

The constant force of mortality interpolation method sets μ_{x+s} equal to a constant for x an integral age and $0 < s \le 1$. Since $p_x = \exp\left(-\int_0^1 \mu_{x+s}\,\mathrm{d}s\right)$ and $\mu_{x+s} = \mu$ is constant,

$$p_x = e^{-\mu} \tag{7.8}$$
$$\mu = -\ln p_x \tag{7.9}$$

Moreover, ${}_sp_x = e^{-\mu s} = (p_x)^s$. In fact, ${}_sp_{x+t}$ is independent of t for $0 \le t \le 1 - s$.

$${}_sp_{x+t} = (p_x)^s \tag{7.10}$$

for any $0 \le t \le 1 - s$. Figure 7.1 shows l_{100+s} and μ_{100+s} for $l_{100} = 1000$ and $q_{100} = 0.45$ if constant force of mortality is assumed.

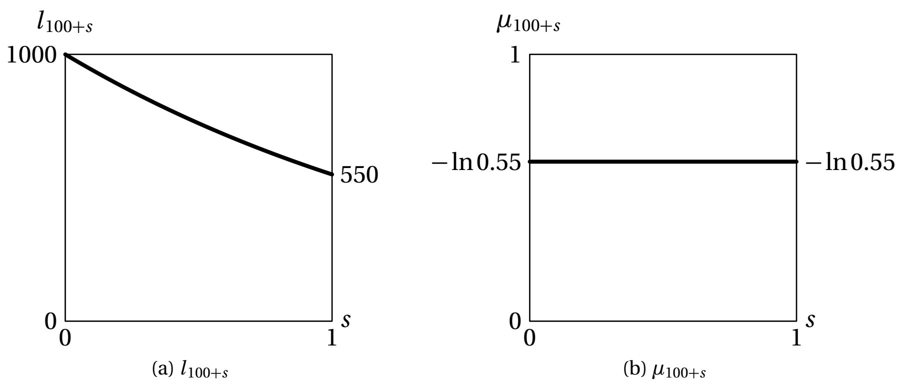


Figure 7.1: Example of constant force of mortality

Contrast constant force of mortality between integral ages to global constant force of mortality, which was introduced in Subsection 4.2.1. The method discussed here allows μ_x to vary for different integers x.

We will now repeat some of the earlier examples but using constant force of mortality.

EXAMPLE 7F You are given:

(i) $q_x = 0.1$
(ii) The force of mortality is constant between integral ages.

Calculate ${}_{1/2}q_{x+1/4}$.

ANSWER:

$${}_{1/2}q_{x+1/4} = 1 - {}_{1/2}p_{x+1/4} = 1 - p_x^{1/2} = 1 - 0.9^{1/2} = 1 - 0.948683 = \boxed{\mathbf{0.051317}}$$ □

EXAMPLE 7G You are given:

(i) $q_x = 0.1$
(ii) $q_{x+1} = 0.2$
(iii) The force of mortality is constant between integral ages.

Calculate $\mathring{e}_{x+0.5:\overline{1}|}$.

ANSWER: We calculate $\int_0^1 {}_tp_{x+0.5}\,dt$. We split this up into two integrals, one from 0 to 0.5 for age x and one from 0.5 to 1 for age $x+1$. The first integral is

$$\int_0^{0.5} {}_tp_{x+0.5}\,dt = \int_0^{0.5} p_x^t\,dt = \int_0^{0.5} 0.9^t\,dt = -\frac{1-0.9^{0.5}}{\ln 0.9} = 0.487058$$

For $t > 0.5$,

$${}_tp_{x+0.5} = {}_{0.5}p_{x+0.5}\;{}_{t-0.5}p_{x+1} = 0.9^{0.5}\;{}_{t-0.5}p_{x+1}$$

so the second integral is

$$0.9^{0.5}\int_{0.5}^{1} {}_{t-0.5}p_{x+1}\,dt = 0.9^{0.5}\int_0^{0.5} 0.8^t\,dt = -\left(0.9^{0.5}\right)\left(\frac{1-0.8^{0.5}}{\ln 0.8}\right) = (0.948683)(0.473116) = 0.448837$$

The answer is $\mathring{e}_{x+0.5:\overline{1}|} = 0.487058 + 0.448837 = \boxed{\mathbf{0.935895}}$. □

Although constant force of mortality is not used as often as UDD, it can be useful for simplifying formulas under certain circumstances. Calculating the expected present value of an insurance where the death benefit within a year follows an exponential pattern (this can happen when the death benefit is the discounted present value of something) may be easier with constant force of mortality than with UDD.

The formulas for this lesson are summarized in Table 7.1.

Exercises

Uniform distribution of death

7.1. [**CAS4-S85:16**] (1 point) Deaths are uniformly distributed between integral ages.

Which of the following represents ${}_{3/4}p_x + \frac{1}{2}\,{}_{1/2}p_x\,\mu_{x+1/2}$?

(A) ${}_{3/4}p_x$ (B) ${}_{3/4}q_x$ (C) ${}_{1/2}p_x$ (D) ${}_{1/2}q_x$ (E) ${}_{1/4}p_x$

7.2. [Based on **150-S88:25**] You are given:

(i) ${}_{0.25}q_{x+0.75} = 3/31$.
(ii) Mortality is uniformly distributed within age x.

Calculate q_x.

Exercises continue on the next page . . .

Table 7.1: Summary of formulas for fractional ages

Function	Uniform distribution of deaths	Constant force of mortality
l_{x+s}	$l_x - s\,d_x$	$l_x\,p_x^s$
${}_sq_x$	$s\,q_x$	$1-p_x^s$
${}_sp_x$	$1-s\,q_x$	p_x^s
${}_sq_{x+t}$	$s\,q_x/(1-t\,q_x)$	$1-p_x^s$
μ_{x+s}	$q_x/(1-s\,q_x)$	$-\ln p_x$
${}_sp_x\,\mu_{x+s}$	q_x	$(p_x^s)(\ln p_x)$
$\mathring{e}_x$	$e_x + 0.5$	
$\mathring{e}_{x:\overline{n}\rceil}$	$e_{x:\overline{n}\rceil} + 0.5\,{}_nq_x$	
$\mathring{e}_{x:\overline{1}\rceil}$	$p_x + 0.5q_x$	

Use the following information for questions 7.3 and 7.4:

You are given:

(i) Deaths are uniformly distributed between integral ages.

(ii) $q_x = 0.10$.

(iii) $q_{x+1} = 0.15$.

7.3. Calculate ${}_{1/2}q_{x+3/4}$.

7.4. Calculate ${}_{0.3|0.5}q_{x+0.4}$.

7.5. You are given:

(i) Deaths are uniformly distributed between integral ages.

(ii) Mortality follows the Illustrative Life Table.

Calculate the median future lifetime for (45.5).

7.6. [**160-F90:5**] You are given:

(i) A survival distribution is defined by

$$l_x = 1000\left(1-\left(\frac{x}{100}\right)^2\right),\ 0 \le x \le 100.$$

(ii) μ_x denotes the actual force of mortality for the survival distribution.

(iii) μ_x^L denotes the approximation of the force of mortality based on the uniform distribution of deaths assumption for l_x, $50 \le x < 51$.

Calculate $\mu_{50.25} - \mu_{50.25}^L$.

(A) -0.00016 (B) -0.00007 (C) 0 (D) 0.00007 (E) 0.00016

Exercises continue on the next page ...

7.7. A survival distribution is defined by

(i) $S_0(k) = 1/(1+0.01k)^4$ for k a non-negative integer.
(ii) Deaths are uniformly distributed between integral ages.

Calculate ${}_{0.4}q_{20.2}$.

7.8. [Based on**150-S89:15**] You are given:

(i) Deaths are uniformly distributed over each year of age.
(ii)

x	l_x
35	100
36	99
37	96
38	92
39	87

Which of the following are true?

I. ${}_{1|2}q_{36} = 0.091$

II. $\mu_{37.5} = 0.043$

III. ${}_{0.33}q_{38.5} = 0.021$

(A) I and II only (B) I and III only (C) II and III only (D) I, II and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

7.9. [**150-82-94:5**] You are given:

(i) Deaths are uniformly distributed over each year of age.
(ii) ${}_{0.75}p_x = 0.25$.

Which of the following are true?

I. ${}_{0.25}q_{x+0.5} = 0.5$

II. ${}_{0.5}q_x = 0.5$

III. $\mu_{x+0.5} = 0.5$

(A) I and II only (B) I and III only (C) II and III only (D) I, II and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

7.10. [**3-S00:12**] For a certain mortality table, you are given:

(i) $\mu_{80.5} = 0.0202$
(ii) $\mu_{81.5} = 0.0408$
(iii) $\mu_{82.5} = 0.0619$
(iv) Deaths are uniformly distributed between integral ages.

Calculate the probability that a person age 80.5 will die within two years.

(A) 0.0782 (B) 0.0785 (C) 0.0790 (D) 0.0796 (E) 0.0800

Exercises continue on the next page . . .

7.11. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) $q_x = 0.1$.
(iii) $q_{x+1} = 0.3$.

Calculate $\mathring{e}_{x+0.7:\overline{1}|}$.

7.12. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) $q_{45} = 0.01$.
(iii) $q_{46} = 0.011$.

Calculate $\operatorname{Var}\Big(\min\big(T_{45}, 2\big)\Big)$.

7.13. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) ${}_{10}p_x = 0.2$.

Calculate $\mathring{e}_{x:\overline{10}|} - e_{x:\overline{10}|}$.

7.14. [**4-F86:21**] You are given:

(i) $q_{60} = 0.020$
(ii) $q_{61} = 0.022$
(iii) Deaths are uniformly distributed over each year of age.

Calculate $\mathring{e}_{60:\overline{1.5}|}$.

(A) 1.447 (B) 1.457 (C) 1.467 (D) 1.477 (E) 1.487

7.15. [**150-F89:21**] You are given:

(i) $q_{70} = 0.040$
(ii) $q_{71} = 0.044$
(iii) Deaths are uniformly distributed over each year of age.

Calculate $\mathring{e}_{70:\overline{1.5}|}$.

(A) 1.435 (B) 1.445 (C) 1.455 (D) 1.465 (E) 1.475

7.16. [**3-S01:33**] For a 4-year college, you are given the following probabilities for dropout from all causes:

$$q_0 = 0.15$$
$$q_1 = 0.10$$
$$q_2 = 0.05$$
$$q_3 = 0.01$$

Dropouts are uniformly distributed over each year.

Compute the temporary 1.5-year complete expected college lifetime of a student entering the second year, $\mathring{e}_{1:\overline{1.5}|}$.

(A) 1.25 (B) 1.30 (C) 1.35 (D) 1.40 (E) 1.45

Exercises continue on the next page . . .

7.17. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) $\mathring{e}_{x+0.5:\overline{0.5}|} = 5/12$.

Calculate q_x.

7.18. You are given:

(i) Deaths are uniformly distributed over each year of age.
(ii) $\mathring{e}_{55.2:\overline{0.4}|} = 0.396$.

Calculate $\mu_{55.2}$.

7.19. [**150-S87:21**] You are given:

(i) $d_x = k$ for $x = 0, 1, 2, \ldots, \omega - 1$
(ii) $\mathring{e}_{20:\overline{20}|} = 18$
(iii) Deaths are uniformly distributed over each year of age.

Calculate ${}_{30|10}q_{30}$.

(A) 0.111 (B) 0.125 (C) 0.143 (D) 0.167 (E) 0.200

7.20. [**150-S89:24**] You are given:

(i) Deaths are uniformly distributed over each year of age.
(ii) $\mu_{45.5} = 0.5$

Calculate $\mathring{e}_{45:\overline{1}|}$.

(A) 0.4 (B) 0.5 (C) 0.6 (D) 0.7 (E) 0.8

7.21. [**CAS3-S04:10**] 4,000 people age (30) each pay an amount, P, into a fund. Immediately after the 1,000$^{\text{th}}$ death, the fund will be dissolved and each of the survivors will be paid \$50,000.

- Mortality follows the Illustrative Life Table, using linear interpolation at fractional ages.
- $i = 12\%$

Calculate P.

(A) Less than 515
(B) At least 515, but less than 525
(C) At least 525, but less than 535
(D) At least 535, but less than 545
(E) At least 545

Exercises continue on the next page . . .

7.22. [**CAS3-S04:10**] 4,000 people age (30) each pay an amount, P, into a fund. Immediately after the 1,000th death, the fund will be dissolved and each of the survivors will be paid $50,000.

- Mortality follows the Illustrative Life Table, using linear interpolation at fractional ages.
- $i = 12\%$

Calculate P.

(A) Less than 515
(B) At least 515, but less than 525
(C) At least 525, but less than 535
(D) At least 535, but less than 545
(E) At least 545

Constant force of mortality

7.23. [**160-F87:5**] Based on given values of l_x and l_{x+1}, ${}_{1/4}p_{x+1/4} = 49/50$ under the assumption of constant force of mortality.

Calculate ${}_{1/4}p_{x+1/4}$ under the uniform distribution of deaths hypothesis.

(A) 0.9799 (B) 0.9800 (C) 0.9801 (D) 0.9802 (E) 0.9803

7.24. [**160-S89:5**] A mortality study is conducted for the age interval $(x, x+1]$.

If a constant force of mortality applies over the interval, ${}_{0.25}q_{x+0.1} = 0.05$.

Calculate ${}_{0.25}q_{x+0.1}$ assuming a uniform distribution of deaths applies over the interval.

(A) 0.044 (B) 0.047 (C) 0.050 (D) 0.053 (E) 0.056

7.25. [**150-F89:29**] You are given that $q_x = 0.25$.

Based on the constant force of mortality assumption, the force of mortality is μ^A_{x+s}, $0 < s < 1$.

Based on the uniform distribution of deaths assumption, the force of mortality is μ^B_{x+s}, $0 < s < 1$.

Calculate the smallest s such that $\mu^B_{x+s} \geq \mu^A_{x+s}$.

(A) 0.4523 (B) 0.4758 (C) 0.5001 (D) 0.5242 (E) 0.5477

7.26. [**160-S91:4**] From a population mortality study, you are given:

(i) Within each age interval, $(x+k, x+k+1]$, the force of mortality, μ_{x+k}, is constant.

(ii)

k	$e^{-\mu_{x+k}}$	$\dfrac{1-e^{-\mu_{x+k}}}{\mu_{x+k}}$
0	0.98	0.99
1	0.96	0.98

Calculate $\mathring{e}_{x:\overline{2}|}$, the expected lifetime in years over $(x, x+2]$.

(A) 1.92 (B) 1.94 (C) 1.95 (D) 1.96 (E) 1.97

Exercises continue on the next page ...

7.27. You are given:

(i) $q_{80} = 0.1$

(ii) $q_{81} = 0.2$

(iii) The force of mortality is constant between integral ages.

Calculate $\mathring{e}_{80.5:\overline{1}|}$.

(A) 0.93 (B) 0.94 (C) 0.95 (D) 0.96 (E) 0.97

7.28. [**3-S01:27**] An actuary is modeling the mortality of a group of 1000 people, each age 95, for the next three years.

The actuary starts by calculating the expected number of survivors at each integral age by

$$l_{95+k} = 1000\,{}_k p_{95}, \qquad k = 1, 2, 3$$

The actuary subsequently calculates the expected number of survivors at the middle of each year using the assumption that deaths are uniformly distributed over each year of age.

This is the result of the actuary's model:

Age	Survivors
95	1000
95.5	800
96	600
96.5	480
97	—
97.5	288
98	—

The actuary decides to change his assumption for mortality at fractional ages to the constant force assumption. He retains his original assumption for each ${}_k p_{95}$.

Calculate the revised expected number of survivors at age 97.5

(A) 270 (B) 273 (C) 276 (D) 279 (E) 282

7.29. [**M-F06:16**] You are given the following information on participants entering a 2-year program for treatment of a disease:

(i) Only 10% survive to the end of the second year.

(ii) The force of mortality is constant within each year.

(iii) The force of mortality for year 2 is three times the force of mortality for year 1.

Calculate the probability that a participant who survives to the end of month 3 dies by the end of month 21.

(A) 0.61 (B) 0.66 (C) 0.71 (D) 0.75 (E) 0.82

Exercises continue on the next page . . .

7.30. [**Sample Question #267**] You are given:

(i) $\mu_x = \sqrt{\dfrac{1}{80-x}}, \quad 0 \le x \le 809$

(ii) F is the exact value of $S_0(10.5)$.

(iii) G is the value of $S_0(10.5)$ using the constant force assumption for interpolation between ages 10 and 11.

Calculate $F-G$.

(A) −0.01083 (B) −0.00005 (C) 0 (D) 0.00003 (E) 0.00172

Additional old CAS Exam 3/3L questions: S05:31, F05:13, S06:13, F06:13, S07:24, S08:16, S09:3, F09:3, S10:4, F10:3, S11:3

Solutions

7.1. In the second summand, ${}_{1/2}p_x\,\mu_{x+1/2}$ is the density function, which is the constant q_x under UDD. The first summand ${}_{3/4}p_x = 1-\frac{3}{4}q_x$. So the sum is $1-\frac{1}{4}q_x$, or $\boxed{{}_{1/4}p_x}$. **(E)**

7.2. Using equation (7.3),

$$\begin{aligned}\frac{3}{31} &= {}_{0.25}q_{x+0.75} = \frac{0.25q_x}{1-0.75q_x}\\ \frac{3}{31}-\frac{2.25}{31}q_x &= 0.25q_x\\ \frac{3}{31} &= \frac{10}{31}q_x\\ q_x &= \boxed{0.3}\end{aligned}$$

7.3. We calculate the probability that $(x+\frac{3}{4})$ survives for half a year. Since the duration crosses an integer boundary, we break the period up into two quarters of a year. The probability of $(x+3/4)$ surviving for 0.25 years is, by equation (7.3),

$${}_{1/4}p_{x+3/4} = \frac{1-0.10}{1-0.75(0.10)} = \frac{0.9}{0.925}$$

The probability of $(x+1)$ surviving to $x+1.25$ is

$${}_{1/4}p_{x+1} = 1-0.25(0.15) = 0.9625$$

The answer to the question is then the complement of the product of these two numbers:

$${}_{1/2}q_{x+3/4} = 1-{}_{1/2}p_{x+3/4} = 1-{}_{1/4}p_{x+3/4}\,{}_{1/4}p_{x+1} = 1-\left(\frac{0.9}{0.925}\right)(0.9625) = \boxed{0.06351}$$

Alternatively, you could build a life table starting at age x, with $l_x = 1$. Then $l_{x+1} = (1-0.1) = 0.9$ and $l_{x+2} = 0.9(1-0.15) = 0.765$. Under UDD, l_x at fractional ages is obtained by linear interpolation, so

$$\begin{aligned}l_{x+0.75} &= 0.75(0.9)+0.25(1) = 0.925\\ l_{x+1.25} &= 0.25(0.765)+0.75(0.9) = 0.86625\\ {}_{1/2}p_{3/4} &= \frac{l_{x+1.25}}{l_{x+0.75}} = \frac{0.86625}{0.925} = 0.93649\\ {}_{1/2}q_{3/4} &= 1-{}_{1/2}p_{3/4} = 1-0.93649 = \boxed{0.06351}\end{aligned}$$

7.4. ${}_{0.3|0.5}q_{x+0.4}$ is ${}_{0.3}p_{x+0.4} - {}_{0.8}p_{x+0.4}$. The first summand is

$$ {}_{0.3}p_{x+0.4} = \frac{1-0.7q_x}{1-0.4q_x} = \frac{1-0.07}{1-0.04} = \frac{93}{96} $$

The probability that $(x+0.4)$ survives to $x+1$ is, by equation (7.3),

$$ {}_{0.6}p_{x+0.4} = \frac{1-0.10}{1-0.04} = \frac{90}{96} $$

and the probability $(x+1)$ survives to $x+1.2$ is

$$ {}_{0.2}p_{x+1} = 1-0.2q_{x+1} = 1-0.2(0.15) = 0.97 $$

So

$$ {}_{0.3|0.5}q_{x+0.4} = \frac{93}{96} - \left(\frac{90}{96}\right)(0.97) = \boxed{\mathbf{0.059375}} $$

Alternatively, you could use the life table from the solution to the last question, and linearly interpolate:

$$ \begin{aligned} l_{x+0.4} &= 0.4(0.9)+0.6(1) = 0.96 \\ l_{x+0.7} &= 0.7(0.9)+0.3(1) = 0.93 \\ l_{x+1.2} &= 0.2(0.765)+0.8(0.9) = 0.873 \\ {}_{0.3|0.5}q_{x+0.4} &= \frac{0.93-0.873}{0.96} = \boxed{\mathbf{0.059375}} \end{aligned} $$

7.5. Under uniform distribution of deaths between integral ages, $l_{x+0.5} = \frac{1}{2}(l_x + l_{x+1})$, since the survival function is a straight line between two integral ages. Therefore, $l_{45.5} = \frac{1}{2}(9{,}164{,}051 + 9{,}127{,}426) = 9{,}145{,}738.5$. Median future lifetime occurs when $l_x = \frac{1}{2}(9{,}145{,}738.5) = 4{,}572{,}869$. This happens between ages 77 and 78. We interpolate between the ages to get the exact median:

$$ \begin{aligned} l_{77} - s(l_{77} - l_{78}) &= 4{,}572{,}869 \\ 4{,}828{,}182 - s(4{,}828{,}182 - 4{,}530{,}360) &= 4{,}572{,}869 \\ 4{,}828{,}182 - 297{,}822s &= 4{,}572{,}869 \\ s = \frac{4{,}828{,}182 - 4{,}572{,}869}{297{,}822} = \frac{255{,}313}{297{,}822} &= 0.8573 \end{aligned} $$

So the median age at death is 77.8573, and median future lifetime is $77.8573 - 45.5 = \boxed{\mathbf{32.3573}}$.

7.6. ${}_xp_0 = \frac{l_x}{l_0} = 1 - \left(\frac{x}{100}\right)^2$. The force of mortality is calculated as the negative derivative of $\ln {}_xp_0$:

$$ \begin{aligned} \mu_x &= -\frac{\mathrm{d}\ln {}_xp_0}{\mathrm{d}x} = \frac{2\left(\frac{x}{100}\right)\left(\frac{1}{100}\right)}{1-\left(\frac{x}{100}\right)^2} = \frac{2x}{100^2 - x^2} \\ \mu_{50.25} &= \frac{100.5}{100^2 - 50.25^2} = 0.0134449 \end{aligned} $$

For UDD, we need to calculate q_{50}.

$$ \begin{aligned} p_{50} &= \frac{l_{51}}{l_{50}} = \frac{1-0.51^2}{1-0.50^2} = 0.986533 \\ q_{50} &= 1 - 0.986533 = 0.013467 \end{aligned} $$

so under UDD,

$$ \mu^L_{50.25} = \frac{q_{50}}{1-0.25q_{50}} = \frac{0.013467}{1-0.25(0.013467)} = 0.013512. $$

The difference between $\mu_{50.25}$ and $\mu^L_{50.25}$ is $0.013445 - 0.013512 = \boxed{\mathbf{-0.000067}}$. **(B)**

7.7. $S_0(20) = 1/1.2^4$ and $S_0(21) = 1/1.21^4$, so $q_{20} = 1 - (1.2/1.21)^4 = 0.03265$. Then

$$ {}_{0.4}q_{20.2} = \frac{0.4q_{20}}{1-0.2q_{20}} = \frac{0.4(0.03265)}{1-0.2(0.03265)} = \boxed{\mathbf{0.01315}} $$

7.8.

I. Calculate ${}_{1|2}q_{36}$.

$$ {}_{1|2}q_{36} = \frac{{}_2d_{37}}{l_{36}} = \frac{96-87}{99} = 0.09091 \quad \checkmark $$

This statement does not require uniform distribution of deaths.

II. By equation (7.5),

$$ \mu_{37.5} = \frac{q_{37}}{1-0.5q_{37}} = \frac{4/96}{1-2/96} = \frac{4}{94} = 0.042553 \quad \checkmark $$

III. Calculate ${}_{0.33}q_{38.5}$.

$$ {}_{0.33}q_{38.5} = \frac{{}_{0.33}d_{38.5}}{l_{38.5}} = \frac{(0.33)(5)}{89.5} = 0.018436 \quad \times $$

I can't figure out what mistake you'd have to make to get 0.021. **(A)**

7.9. First calculate q_x.

$$ \begin{aligned} 1 - 0.75q_x &= 0.25 \\ q_x &= 1 \end{aligned} $$

Then by equation (7.3), ${}_{0.25}q_{x+0.5} = 0.25/(1-0.5) = 0.5$, making I true.
By equation (7.2), ${}_{0.5}q_x = 0.5q_x = 0.5$, making II true.
By equation (7.5), $\mu_{x+0.5} = 1/(1-0.5) = 2$, making III false. **(A)**

7.10. We use equation (7.5) to back out q_x for each age.

$$ \begin{aligned} \mu_{x+0.5} = \frac{q_x}{1-0.5q_x} &\Rightarrow q_x = \frac{\mu_{x+0.5}}{1+0.5\mu_{x+0.5}} \\ q_{80} &= \frac{0.0202}{1.0101} = 0.02 \\ q_{81} &= \frac{0.0408}{1.0204} = 0.04 \\ q_{82} &= \frac{0.0619}{1.03095} = 0.06 \end{aligned} $$

Then by equation (7.3), ${}_{0.5}p_{80.5} = 0.98/0.99$. $p_{81} = 0.96$, and ${}_{0.5}p_{82} = 1 - 0.5(0.06) = 0.97$. Therefore

$$ {}_2q_{80.5} = 1 - \left(\frac{0.98}{0.99}\right)(0.96)(0.97) = \boxed{\mathbf{0.0782}} \qquad \textbf{(A)} $$

7.11. To do this algebraically, we split the group into those who die within 0.3 years, those who die between 0.3 and 1 years, and those who survive one year. Under UDD, those who die will die at the midpoint of the interval (assuming the interval doesn't cross an integral age), so we have

Group	Survival time	Probability of group	Average survival time
I	$(0,0.3]$	$1-{}_{0.3}p_{x+0.7}$	0.15
II	$(0.3,1]$	${}_{0.3}p_{x+0.7}-{}_{1}p_{x+0.7}$	0.65
III	$(1,\infty)$	${}_{1}p_{x+0.7}$	1

We calculate the required probabilities.

$$\begin{aligned}
{}_{0.3}p_{x+0.7} &= \frac{0.9}{0.93} = 0.967742\\
{}_{1}p_{x+0.7} &= \frac{0.9}{0.93}\bigl(1-0.7(0.3)\bigr) = 0.764516\\
1-{}_{0.3}p_{x+0.7} &= 1-0.967742 = 0.032258\\
{}_{0.3}p_{x+0.7}-{}_{1}p_{x+0.7} &= 0.967742-0.764516 = 0.203226\\
\mathring{e}_{x+0.7:\overline{1}|} &= 0.032258(0.15)+0.203226(0.65)+0.764516(1) = \boxed{\mathbf{0.901452}}
\end{aligned}$$

Alternatively, we can use trapezoids. We already know from the above solution that the heights of the first trapezoid are 1 and 0.967742, and the heights of the second trapezoid are 0.967742 and 0.764516. So the sum of the area of the two trapezoids is

$$\begin{aligned}
\mathring{e}_{x+0.7:\overline{1}|} &= (0.3)(0.5)(1+0.967742)+(0.7)(0.5)(0.967742+0.764516)\\
&= 0.295161+0.606290 = \boxed{\mathbf{0.901451}}
\end{aligned}$$

7.12. For the expected value, we'll use the recursive formula. (The trapezoidal rule could also be used.)

$$\begin{aligned}
\mathring{e}_{45:\overline{2}|} &= \mathring{e}_{45:\overline{1}|} + p_{45}\,\mathring{e}_{46:\overline{1}|}\\
&= (1-0.005)+0.99(1-0.0055)\\
&= 1.979555
\end{aligned}$$

We'll use equation (5.7)to calculate the second moment.

$$\begin{aligned}
\mathbf{E}[\min(T_{45},2)^2] &= 2\int_0^2 t\,{}_tp_x\,dt\\
&= 2\left(\int_0^1 t(1-0.01t)\,dt + \int_1^2 t(0.99)\bigl(1-0.011(t-1)\bigr)\,dt\right)\\
&= 2\left(\frac{1}{2}-0.01\left(\frac{1}{3}\right)+0.99\left(\frac{(1.011)(2^2-1^2)}{2}-0.011\left(\frac{2^3-1^3}{3}\right)\right)\right)\\
&= 2(0.496667+1.475925) = 3.94518
\end{aligned}$$

So the variance is $3.94518-1.979555^2 = \boxed{\mathbf{0.02654}}$.

7.13. As discussed on page 127, by equation (7.7), the difference is

$$\frac{1}{2}\,{}_{10}q_x = \frac{1}{2}(1-0.2) = \boxed{\mathbf{0.4}}$$

7.14. Those who die in the first year survive $\frac{1}{2}$ year on the average and those who die in the first half of the second year survive 1.25 years on the average, so we have

$$p_{60} = 0.98$$
$$_{1.5}p_{60} = 0.98\big(1-0.5(0.022)\big) = 0.96922$$
$$\mathring{e}_{60:\overline{1.5}|} = 0.5(0.02)+1.25(0.98-0.96922)+1.5(0.96922) = \boxed{\mathbf{1.477305}} \qquad \textbf{(D)}$$

Alternatively, we use the trapezoidal method. The first trapezoid has heights 1 and $p_{60} = 0.98$ and width 1. The second trapezoid has heights $p_{60} = 0.98$ and $_{1.5}p_{60} = 0.96922$ and width 1/2.

$$\mathring{e}_{60:\overline{1.5}|} = \frac{1}{2}(1+0.98)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(0.98+0.96922)$$
$$= \boxed{\mathbf{1.477305}} \qquad \textbf{(D)}$$

7.15. $p_{70} = 1-0.040 = 0.96$, $_2p_{70} = (0.96)(0.956) = 0.91776$, and by linear interpolation, $_{1.5}p_{70} = 0.5(0.96+0.91776) = 0.93888$. Those who die in the first year survive 0.5 years on the average and those who die in the first half of the second year survive 1.25 years on the average. So

$$\mathring{e}_{70:\overline{1.5}|} = 0.5(0.04)+1.25(0.96-0.93888)+1.5(0.93888) = \boxed{\mathbf{1.45472}} \qquad \textbf{(C)}$$

Alternatively, we can use the trapezoidal method. The first year's trapezoid has heights 1 and 0.96 and width 1 and the second year's trapezoid has heights 0.96 and 0.93888 and width 1/2, so

$$\mathring{e}_{70:\overline{1.5}|} = 0.5(1+0.96)+0.5(0.5)(0.96+0.93888) = \boxed{\mathbf{1.45472}} \qquad \textbf{(C)}$$

7.16. First we calculate $_tp_1$ for $t = 1, 2$.

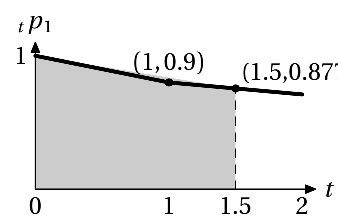
$$p_1 = 1-q_1 = 0.90$$
$$_2p_1 = (1-q_1)(1-q_2) = (0.90)(0.95) = 0.855$$

By linear interpolation, $_{1.5}p_1 = (0.5)(0.9+0.855) = 0.8775$.

The algebraic method splits the students into three groups: first year dropouts, second year (up to time 1.5) dropouts, and survivors. In each dropout group survival on the average is to the midpoint (0.5 years for the first group, 1.25 years for the second group) and survivors survive 1.5 years. Therefore

$$\mathring{e}_{1:\overline{1.5}|} = 0.10(0.5)+(0.90-0.8775)(1.25)+0.8775(1.5) = \boxed{\mathbf{1.394375}} \qquad \textbf{(D)}$$

Alternatively, we could sum the two trapezoids making up the shaded area at the right.

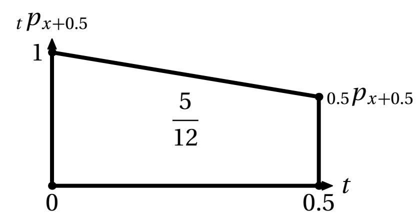


$$\mathring{e}_{1:\overline{1.5}|} = (1)(0.5)(1+0.9)+(0.5)(0.5)(0.90+0.8775)$$
$$= 0.95+0.444375 = \boxed{\mathbf{1.394375}} \qquad \textbf{(D)}$$

7.17. Those who die survive 0.25 years on the average and survivors survive 0.5 years, so we have

$$\begin{aligned}
0.25\,{}_{0.5}q_{x+0.5} + 0.5\,{}_{0.5}p_{x+0.5} &= \tfrac{5}{12}\\
0.25\left(\frac{0.5q_x}{1-0.5q_x}\right) + 0.5\left(\frac{1-q_x}{1-0.5q_x}\right) &= \tfrac{5}{12}\\
0.125q_x + 0.5 - 0.5q_x &= \tfrac{5}{12} - \tfrac{5}{24}q_x\\
\frac{1}{2} - \frac{5}{12} &= \left(-\frac{5}{24} + \frac{1}{2} - \frac{1}{8}\right) q_x\\
\frac{1}{12} &= \frac{q_x}{6}\\
q_x &= \boxed{\tfrac{1}{2}}
\end{aligned}$$

Alternatively, complete life expectancy is the area of the trapezoid shown on the right, so



$$\frac{5}{12} = 0.5(0.5)(1 + {}_{0.5}p_{x+0.5})$$

Then ${}_{0.5}p_{x+0.5} = \frac{2}{3}$, from which it follows

$$\begin{aligned}
\frac{2}{3} &= \frac{1-q_x}{1-\frac{1}{2}q_x}\\
q_x &= \boxed{\tfrac{1}{2}}
\end{aligned}$$

7.18. Survivors live 0.4 years and those who die live 0.2 years on the average, so

$$0.396 = 0.4\,{}_{0.4}p_{55.2} + 0.2\,{}_{0.4}q_{55.2}$$

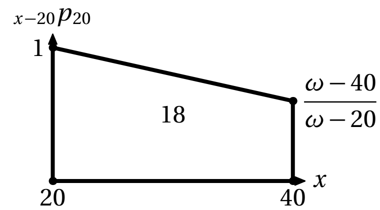
Using the formula ${}_{0.4}q_{55.2} = 0.4q_{55}/(1-0.2q_{55})$ (equation (7.3)), we have

$$\begin{aligned}
0.4\left(\frac{1-0.6q_{55}}{1-0.2q_{55}}\right) + 0.2\left(\frac{0.4q_{55}}{1-0.2q_{55}}\right) &= 0.396\\
0.4 - 0.24q_{55} + 0.08q_{55} &= 0.396 - 0.0792q_{55}\\
0.0808q_{55} &= 0.004\\
q_{55} = \frac{0.004}{0.0808} &= 0.0495\\
\mu_{55.2} = \frac{q_{55}}{1-0.2q_{55}} = \frac{0.0495}{1-0.2(0.0495)} &= \boxed{\mathbf{0.05}}
\end{aligned}$$

7.19. Since d_x is constant for all x and deaths are uniformly distributed within each year of age, mortality is uniform globally. We back out ω using equation (5.10), $\mathring{e}_{x:\overline{n}|} = {}_np_x(n) + {}_nq_x(n/2)$:

$$\begin{aligned}
10\,{}_{20}q_{20} + 20\,{}_{20}p_{20} &= 18\\
10\left(\frac{20}{\omega-20}\right) + 20\left(\frac{\omega-40}{\omega-20}\right) &= 18\\
200 + 20\omega - 800 &= 18\omega - 360\\
2\omega &= 240\\
\omega &= 120
\end{aligned}$$

Alternatively, we can back out ω using the trapezoidal rule. Complete life expectancy is the area of the trapezoid shown to the right.



$$\begin{aligned}
\mathring{e}_{20:\overline{20}|} = 18 &= (20)(0.5)\left(1+\frac{\omega-40}{\omega-20}\right)\\
0.8 &= \frac{\omega-40}{\omega-20}\\
0.8\omega-16 &= \omega-40\\
0.2\omega &= 24\\
\omega &= 120
\end{aligned}$$

Once we have ω, we compute

$$_{30|10}q_{30} = \frac{10}{\omega-30} = \frac{10}{90} = \boxed{\mathbf{0.1111}} \qquad \textbf{(A)}$$

7.20. We use equation (7.5) to obtain

$$\begin{aligned}
0.5 &= \frac{q_x}{1-0.5q_x}\\
q_x &= 0.4
\end{aligned}$$

Then $\mathring{e}_{45:\overline{1}|} = 0.5\big(1+(1-0.4)\big) = \boxed{\mathbf{0.8}}$. **(E)**

7.21. According to the Illustrative Life Table, $l_{30} = 9{,}501{,}381$, so we are looking for the age x such that $l_x = 0.75(9{,}501{,}381) = 7{,}126{,}036$. This is between 67 and 68. Using linear interpolation, since $l_{67} = 7{,}201{,}635$ and $l_{68} = 7{,}018{,}432$, we have

$$x = 67 + \frac{7{,}201{,}635 - 7{,}126{,}036}{7{,}201{,}635 - 7{,}018{,}432} = 67.4127$$

This is 37.4127 years into the future. $\frac{3}{4}$ of the people collect 50,000. We need $50{,}000\left(\frac{3}{4}\right)\left(\frac{1}{1.12^{37.4127}}\right) = \boxed{\mathbf{540.32}}$ per person. **(D)**

7.22. According to the Illustrative Life Table, $l_{30} = 9{,}501{,}381$, so we are looking for the age x such that $l_x = 0.75(9{,}501{,}381) = 7{,}126{,}036$. This is between 67 and 68. Using linear interpolation, since $l_{67} = 7{,}201{,}635$ and $l_{68} = 7{,}018{,}432$, we have

$$x = 67 + \frac{7{,}201{,}635 - 7{,}126{,}036}{7{,}201{,}635 - 7{,}018{,}432} = 67.4127$$

This is 37.4127 years into the future. $\frac{3}{4}$ of the people collect 50,000. We need $50{,}000\left(\frac{3}{4}\right)\left(\frac{1}{1.12^{37.4127}}\right) = \boxed{\mathbf{540.32}}$ per person. **(D)**

7.23. Under constant force, $_sp_{x+t} = p_x^s$, so $p_x = {}_{1/4}p_{x+1/4}^4 = 0.98^4 = 0.922368$ and $q_x = 1 - 0.922368 = 0.077632$. Under uniform distribution of deaths,

$$\begin{aligned}
{}_{1/4}p_{x+1/4} &= 1 - \frac{(1/4)q_x}{1-(1/4)q_x}\\
&= 1 - \frac{(1/4)(0.077632)}{1-(1/4)(0.077632)}\\
&= 1 - 0.019792 = \boxed{\mathbf{0.980208}} \qquad \textbf{(D)}
\end{aligned}$$

7.24. Under constant force, ${}_s p_{x+t} = p_x^s$, so $p_x = 0.95^4 = 0.814506$, $q_x = 1 - 0.814506 = 0.185494$. Then under a uniform assumption,

$$ {}_{0.25}q_{x+0.1} = \frac{0.25q_x}{1-0.1q_x} = \frac{(0.25)(0.185494)}{1-0.1(0.185494)} = \boxed{\mathbf{0.047250}} \qquad \textbf{(B)} $$

7.25. Using constant force, μ^A is a constant equal to $-\ln p_x = -\ln 0.75 = 0.2877$. Then

$$ \begin{aligned} \mu_{x+s}^B &= \frac{q_x}{1-sq_x} = 0.2877 \\ \frac{0.25}{1-0.25s} &= 0.2877 \\ 0.2877 - 0.25(0.2877)s &= 0.25 \\ s &= \frac{0.2877-0.25}{(0.25)(0.2877)} = \boxed{\mathbf{0.5242}} \qquad \textbf{(D)} \end{aligned} $$

7.26. We integrate ${}_t p_x$ from 0 to 2. Between 0 and 1, ${}_t p_x = e^{-t\mu_x}$.

$$ \int_0^1 e^{-t\mu_x}\,\mathrm{d}t = \frac{1-e^{-\mu_x}}{\mu_x} = 0.99 $$

Between 1 and 2, ${}_t p_x = p_x \, {}_{t-1}p_{x+1} = 0.98e^{-(t-1)\mu_{x+1}}$.

$$ \int_1^2 e^{-(t-1)\mu_{x+1}}\,\mathrm{d}t = \frac{1-e^{-\mu_{x+1}}}{\mu_{x+1}} = 0.98 $$

So the answer is $0.99 + 0.98(0.98) = \boxed{\mathbf{1.9504}}$. **(C)**

7.27.

$$ \begin{aligned} \mathring{e}_{80.5:\overline{1}|} &= \mathring{e}_{80.5:\overline{0.5}|} + {}_{0.5}p_{80.5}\,\mathring{e}_{81:\overline{0.5}|} \\ &= \frac{\int_{0.5}^1 0.9^t\,\mathrm{d}t}{0.9^{0.5}} + 0.9^{0.5}\int_0^{0.5} 0.8^t\,\mathrm{d}t \\ &= \frac{0.9^{0.5}-1}{\ln 0.9} + \left(0.9^{0.5}\right)\frac{0.8^{0.5}-1}{\ln 0.8} \\ &= 0.487058 + (0.948683)(0.473116) = \boxed{\mathbf{0.93590}} \qquad \textbf{(B)} \end{aligned} $$

7.28. Under uniform distribution, the numbers of deaths in each half of the year are equal, so if 120 deaths occurred in the first half of $x = 96$, then 120 occurred in the second half, and $l_{97} = 480 - 120 = 360$. Then if ${}_{0.5}q_{97} = (360-288)/360 = 0.2$, then $q_{97} = 2\,{}_{0.5}q_{97} = 0.4$, so $p_{97} = 0.6$. Under constant force, ${}_{1/2}p_{97} = p_{97}^{0.5} = \sqrt{0.6}$. The answer is $360\sqrt{0.6} = \boxed{\mathbf{278.8548}}$. **(D)**

7.29. Let μ be the force of mortality in year 1. Then 10% survivorship means

$$ \begin{aligned} e^{-\mu-3\mu} &= 0.1 \\ e^{-4\mu} &= 0.1 \end{aligned} $$

The probability of survival 21 months given survival 3 months is the probability of survival 9 months after month 3, or $e^{-(3/4)\mu}$, times the probability of survival another 9 months given survival 1 year, or $e^{-(3/4)3\mu}$, which multiplies to $e^{-3\mu} = (e^{-4\mu})^{3/4} = 0.1^{3/4} = 0.177828$, so the death probability is $1 - 0.177828 = \boxed{\mathbf{0.822172}}$. **(E)**

7.30. The exact value is:

$$F = {}_{10.5}p_0 = \exp\left(-\int_0^{10.5} \mu_x \,\mathrm{d}x\right)$$

$$\int_0^{10.5} (80-x)^{-0.5}\mathrm{d}x = -2(80-x)^{0.5}\Big|_0^{10.5}$$

$$= -2\left(69.5^{0.5} - 80^{0.5}\right) = 1.215212$$

$${}_{10.5}p_0 = e^{-1.215212} = 0.299647$$

To calculate $S_0(10.5)$ with constant force interpolation between 10 and 11, we have ${}_{0.5}p_{10} = p_{10}^{0.5}$, and ${}_{10.5}p_0 = {}_{10}p_0\,{}_{0.5}p_{10}$, so

$$\int_0^{10} (80-x)^{0.5}\mathrm{d}x = -2\left(70^{0.5} - 80^{0.5}\right) = 1.155343$$

$$\int_{10}^{11} (80-x)^{0.5}\mathrm{d}x = -2\left(69^{0.5} - 70^{0.5}\right) = 0.119953$$

$$G = {}_{10.5}p_0 = e^{-1.155343-0.5(0.119953)} = 0.296615$$

Then $F - G = 0.299647 - 0.299615 =$ **0.000032**. **(D)**

Quiz Solutions

7-1. Notice that $\mu_{50.4} = \frac{q_{50}}{1-0.4q_{50}}$ while ${}_{0.6}q_{50.4} = \frac{0.6q_{50}}{1-0.4q_{50}}$, so ${}_{0.6}q_{50.4} = 0.6(0.01) =$ **0.006**

7-2. The algebraic method goes: those who die will survive 0.3 on the average, and those who survive will survive 0.6.

$${}_{0.6}q_{x+0.4} = \frac{0.6(0.1)}{1-0.4(0.1)} = \frac{6}{96}$$

$${}_{0.6}p_{x+0.4} = 1 - \frac{6}{96} = \frac{90}{96}$$

$$\mathring{e}_{x+0.4:\overline{0.6}|} = \frac{6}{96}(0.3) + \frac{90}{96}(0.6) = \frac{55.8}{96} = \boxed{0.58125}$$

The geometric method goes: we need the area of a trapezoid having height 1 at $x+0.4$ and height 90/96 at $x+1$, where 90/96 is ${}_{0.6}p_{x+0.4}$, as calculated above. The width of the trapezoid is 0.6. The answer is therefore $0.5(1+90/96)(0.6) =$ **0.58125**.

7-3. Batteries failing in month 1 survive an average of 0.5 month, those failing in month 2 survive an average of 1.5 months, and those failing in month 3 survive an average of 2.125 months (the average of 2 and 2.25). By linear interpolation, ${}_{2.25}q_0 = 0.25(0.6) + 0.75(0.2) = 0.3$. So we have

$$\mathring{e}_{0:\overline{2.25}|} = q_0(0.5) + {}_{1|}q_0(1.5) + {}_{2|0.25}q_0(2.125) + {}_{2.25}p_0(2.25)$$

$$= (0.05)(0.5) + (0.20-0.05)(1.5) + (0.3-0.2)(2.125) + 0.70(2.25) = \boxed{2.0375}$$

Practice Exam 1

1. You are given:

(i) The following life table.

x	l_x	d_x
50	1000	20
51		
52		35
53		37

(ii) ${}_2q_{52} = 0.07508$.

Determine d_{51}.

(A) 20 (B) 21 (C) 22 (D) 24 (E) 26

2. For a fully discrete 20-year deferred whole life insurance of 1000 on (50), you are given:

(i) Premiums are payable for 20 years.
(ii) The net premium is 12.
(iii) Deaths are uniformly distributed between integral ages.
(iv) $i = 0.1$
(v) ${}_9V = 240$ and ${}_{9.5}V = 266.70$.

Calculate ${}_{10}V$, the benefit reserve at the end of year 10.

(A) 272.75 (B) 280.00 (C) 281.40 (D) 282.28 (E) 282.86

3. For an annual premium 2-year term insurance on (60) with benefit S payable at the end of the year of death, you are given

(i)

t	p_{60+t-1}
1	0.98
2	0.96

(ii) The annual net premium is 25.41.
(iii) $i = 0.05$.

Determine the revised annual net premium if an interest rate of $i = 0.04$ is used.

(A) 25.59 (B) 25.65 (C) 25.70 (D) 25.75 (E) 25.81

4. In a double-decrement model, with decrements (1) and (2), you are given, for all $t > 0$:

(i) ${}_tp_x'^{(1)} = 10/(10+t)$
(ii) ${}_tp_x'^{(2)} = \left(10/(10+t)\right)^3$

Determine $q_x^{(1)}$.

(A) 0.068 (B) 0.074 (C) 0.079 (D) 0.083 (E) 0.091

5. For a special whole life insurance paying benefits at the moment of death, you are given:

(i) If death occurs in the first 10 years, the benefit is the refund of the single benefit premium with interest at a rate of $\delta' = 0.03$.
(ii) If death occurs after the first 10 years, the benefit is 1000 only.
(iii) $\mu_{x+t} = \begin{cases} 0.01 & t \le 10 \\ 0.02 & t > 10 \end{cases}$
(iv) $\delta = 0.06$

Determine the single benefit premium.

(A) 131 (B) 132 (C) 133 (D) 134 (E) 135

6. In a three-decrement model, decrements (1) and (2) are uniformly distributed between integral ages in the independent single decrement tables. Decrement (3) occurs only at the end of a year. You are given:

(i) $l_x^{(\tau)} = 1000$
(ii) $d_x^{(1)} = 90$
(iii) $q_x'^{(2)} = 2q_x'^{(1)}$
(iv) $q_x'^{(3)} = 3q_x'^{(1)}$

Determine $d_x^{(3)}$.

(A) 214 (B) 216 (C) 240 (D) 270 (E) 288

7. For a temporary life annuity-due on (30), you are given:

(i) The annuity makes 20 certain payments.
(ii) The annuity will not make more than 40 payments.
(iii) Mortality follows the Illustrative Life Table.
(iv) $i = 0.06$

Determine the expected present value of the annuity.

(A) 14.79 (B) 15.22 (C) 15.47 (D) 15.63 (E) 16.06

8. An annual premium whole life insurance on (30) pays a benefit of 1 at the end of the year of death.

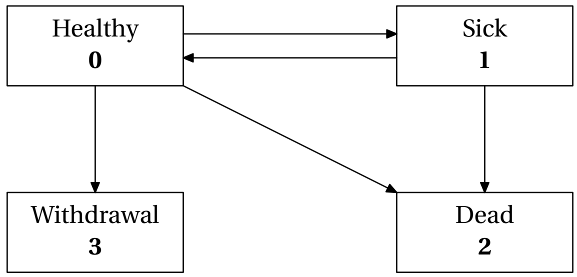
You are given that $A_x = 0.4 + 0.01x$ for $x < 60$.

Calculate the benefit reserve at time 20 for this insurance.

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

Exercises continue on the next page . . .

9. You are given the following Markov chain model for disability income:



Forces of transition are:

$$\mu_x^{01} = 0.002x \qquad \mu^{02} = 0.0001x \qquad \mu^{03} = 0.0004x$$
$$\mu_x^{10} = 0.001x \qquad \mu^{12} = 0.006x$$

Calculate the probability that a healthy individual age 40 ever enters the sick state.

(A) 0.75 (B) 0.80 (C) 0.83 (D) 0.85 (E) 0.95

10. A special 9-year term insurance pays the following benefit at the end of the year of death:

Year of death t	1	2	3	4	5	6	7	8	9
Benefit S_t	1	2	3	4	5	4	3	2	1

$(DA)^{1}_{x:\overline{n}|}$ denotes the expected present value of a decreasing term insurance that pays a benefit of $n+1-k$ at the end of the year if death occurs in year k, $1 \le k \le n$.

You are given the following expected present values for increasing and decreasing term insurances:

n	$(IA)^{1}_{x:\overline{n}|}$	$(DA)^{1}_{x:\overline{n}|}$
4	0.5	0.7
5	0.8	1.0
9	2.3	2.8
10	2.9	3.7

Determine the expected present value of the special term insurance.

(A) 0.7 (B) 0.8 (C) 1.3 (D) 1.4 (E) 1.7

11. You are given:

(i) Automobiles are covered by a 5-year warranty.
(ii) The hazard rate for automobile breakdown is $\mu_t = 2/(10-t)$ for $0 \le t < 10$.

Calculate the expected number of years to breakdown for those automobiles that break down during the warranty period.

(A) 1.67 (B) 1.83 (C) 2.22 (D) 2.92 (E) 3.33

Exercises continue on the next page ...

12. For a 20-year endowment insurance policy of 1000 on (x):

(i) Death benefits are paid at the moment of death.
(ii) Premiums of 46 per year are payable continuously.
(iii) $\mu_{x+t} = 0.02, \quad t \geq 0$
(iv) $\delta = 0.04$

For a portfolio of such policies, the present value of the future loss at issue is estimated using the normal approximation.

Determine the smallest number of policies for which the 95th percentile of the future loss at issue is 0.

(A) 79,300 (B) 89,300 (C) 99,300 (D) 109,300 (E) 119,300

13. An employee's salary rate at exact age 35 is 75,000.

The salary scale is $s_y = 1.03^y$.

Determine the 3-year final average salary for this employee if retirement age is 65.

(A) 171,595 (B) 171,645 (C) 174,149 (D) 174,200 (E) 176,794

14. A life age 90 is subject to mortality following Makeham's law with $A = 0.0005$, $B = 0.0008$, and $c = 1.07$.

Curtate life expectancy for this life is 6.647 years.

Using Woolhouse's formula with three terms, compute complete life expectancy for this life.

(A) 7.118 (B) 7.133 (C) 7.147 (D) 7.161 (E) 7.176

15. For a whole life insurance of 1000 with annual premiums payable for life and benefits paid at the end of the year of death, you are given:

(i) ${}_8V = 210.10$
(ii) ${}_9V = 232.22$
(iii) ${}_{10}V = 255.40$
(iv) $q_{x+8} = q_{x+9}$
(v) $q_{x+10} = 1.1q_{x+9}$
(vi) $i = 0.04$

Determine ${}_{11}V$.

(A) 276.82 (B) 277.58 (C) 278.35 (D) 279.14 (E) 279.69

16. A life age 60 is subject to Gompertz's law with $B = 0.001$ and $c = 1.05$.

Calculate $e_{60:\overline{2}|}$ for this life.

(A) 1.923 (B) 1.928 (C) 1.933 (D) 1.938 (E) 1.943

17. For a fully continuous whole life insurance of 1000 on (x):

(i) The gross premium is paid at an annual rate of 25.
(ii) The variance of future loss is 2,000,000.
(iii) $\delta = 0.06$

Employees are able to obtain this insurance for a 20% discount.

Determine the variance of future loss for insurance sold to employees.

(A) 1,281,533 (B) 1,295,044 (C) 1,771,626 (D) 1,777,778 (E) 1,825,013

Exercises continue on the next page . . .

18. For a continuously increasing continuous annuity on (x) paying at the rate of t per year at time t, you are given:

(i) $\mathbf{E}[T_x] = 52$
(ii) $\operatorname{Var}(T_x) = 822$
(iii) $\delta = 0$

Compute the expected present value of the annuity.

(A) 1302 (B) 1748 (C) 1763 (D) 2518 (E) 2604

19. You are given the following profit test for a 10-year term insurance of 100,000 on (x):

t	${}_{t-1}V$	P	E_t	I_t	bq_{x+t-1}	$p_{x+t-1}\,{}_tV$
0			−350			
1	0	1000	0	60.0	500	447.75
2	450	1000	20	85.8	600	795.20
3	800	1000	20	106.8	700	1092.30
4	1100	1000	20	124.8	800	1289.60
5	1300	1000	20	136.8	900	1412.18
6	1425	1000	20	144.3	1000	1435.50
7	1450	1000	20	145.8	1100	1285.70
8	1300	1000	20	136.8	1200	1037.40
9	1050	1000	20	121.8	1300	641.55
10	650	1000	20	97.8	1400	0.00

Which of the following statements is true?

I. The interest rate used in the calculation is $i = 0.06$.

II. At time 5, the reserve per survivor is 1425.

III. The profit signature in year 3 is 92.81

(A) I and II only (B) I and III only (C) II and III only (D) I, II, and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

20. For a 10-year deferred life annuity-due on (x) of 1000 per year, you are given:

(i) Premiums are payable at the beginning of the first 10 years.
(ii) The annual benefit premium is 800.
(iii) Percent of premium expenses are 5%.
(iv) Expenses incurred with each annuity payment are 10.50.
(v) The gross premium is determined using the equivalence principle.

Determine the gross premium.

(A) 848 (B) 850 (C) 851 (D) 853 (E) 854

Exercises continue on the next page . . .

21. Your company sells whole life insurance policies. At a meeting with the Enterprise Risk Management Committee, it was agreed that you would limit the face amount of the policies sold so that the probability that the present value of the benefit at issue is greater than 1,000,000 is never more than 0.05.

You are given:

(i) The insurance policies pay a benefit equal to the face amount b at the moment of death.
(ii) The force of mortality is $\mu_x = 0.001(1.05^x)$, $x > 0$
(iii) $\delta = 0.06$

Determine the largest face amount b for a policy sold to a purchaser who is age 45.

(A) 1,350,000 (B) 1,400,000 (C) 1,450,000 (D) 1,500,000 (E) 1,550,000

22. For two independent lives (x) and (y), you are given

(i) $\mu_x = 1/(100 - x)$
(ii) $\mu_y = 1/(90 - y)$
(iii) T_{xy} is the future lifetime random variable for the joint status of (x) and (y).

Determine $\mathrm{Var}\left(T_{30:20}\right)$.

(A) 204 (B) 245 (C) 272 (D) 327 (E) 408

23. For a special fully discrete life insurance to (45),

(i) The benefit is 1000 if death occurs before age 65, 500 otherwise.
(ii) Premiums are payable at the beginning of the first 20 years only.
(iii) $i = 0.05$
(iv) $A_{45} = 0.2$
(v) ${}_{20}E_{45} = 0.3$
(vi) $\ddot{a}_{45:\overline{20|}} = 12.6$

Determine the annual benefit premium.

(A) 11.24 (B) 11.90 (C) 13.05 (D) 13.16 (E) 15.87

24. For a fully continuous whole life insurance of 1000 on two independent lives (x) and (y), you are given

(i) Benefits are payable at the moment of the second death.
(ii) Premiums are payable while both are alive.
(iii) $\mu_{x+t} = 0.02$ for all t.
(iv) $\mu_{y+t} = 0.03$ for all t.
(v) $\delta = 0.05$

Determine the annual benefit premium.

(A) 9.57 (B) 12.50 (C) 16.07 (D) 18.37 (E) 21.43

Exercises continue on the next page . . .

25. For a double decrement model with decrements from death (1) and withdrawal (2), you are given:

(i) The following rates in the double-decrement table for (x):

t	Death $q^{(1)}_{x+t-1}$	Withdrawal $q^{(2)}_{x+t-1}$
1	0.003	0.20
2	a	0.15
3	$2a$	0.10

(ii) ${}_3q^{(1)}_x = 0.017985$.

Determine a.

(A) 0.004 (B) 0.005 (C) 0.006 (D) 0.007 (E) 0.008

26. For two lives (50) and (60) with independent future lifetimes:

(i) $\mu_{50+t} = 0.002t, \quad t > 0$

(ii) $\mu_{60+t} = 0.003t, \quad t > 0$

Calculate ${}_{20}q^{\;1}_{50:60} - {}_{20}q^{\;\;\;2}_{50:60}$.

(A) 0.17 (B) 0.18 (C) 0.30 (D) 0.31 (E) 0.37

27. You are given that $\mu_x = 0.002x + 0.005$.

Calculate ${}_{5|}q_{20}$.

(A) 0.015 (B) 0.026 (C) 0.034 (D) 0.042 (E) 0.050

28. For a 30-pay whole life insurance policy of 100,000 on (45), you are given:

(i) Benefits are payable at the end of the year of death.

(ii) Premiums and expenses are payable at the beginning of the year.

(iii) $\ddot{a}_{45} = 14.1121$

(iv) $\ddot{a}_{45:\overline{30}|} = 13.3722$

(v) $i = 0.06$

(vi) Expenses are:

	Per Premium	Per Policy
First Year	40%	200
Renewal Years	10%	r
Settlement		100

(vii) The gross premium determined by the equivalence principle is 1777.98.

Determine r.

(A) 37 (B) 38 (C) 39 (D) 40 (E) 41

Exercises continue on the next page . . .

29. For a special fully discrete whole life insurance on (40), you are given:

(i) The annual benefit premium in the first 20 years is $1000P_{40}$.
(ii) The annual benefit premium changes at age 60.
(iii) The death benefit is 1000 in the first 20 years, after which it is 2000.
(iv) Mortality follows the Illustrative Life Table.
(v) $i = 0.06$

Determine ${}_{21}V$, the benefit reserve for the policy at the end of 21 years.

(A) 282 (B) 286 (C) 292 (D) 296 (E) 300

30. You are given the following yield curve:

$$y_t = \begin{cases} 0.01 + 0.004t & 0 < t \leq 5 \\ 0.02 + 0.002t & 5 \leq t \leq 20 \\ 0.06 & t \geq 20 \end{cases}$$

Calculate the 2-year forward rate on a 10-year zero-coupon bond.

(A) 0.040 (B) 0.044 (C) 0.047 (D) 0.049 (E) 0.052

Solutions to the above questions begin on page 1229.

Appendix A. Solutions to the Practice Exams

Answer Key for Practice Exam 1

1	B	11	C	21	A
2	D	12	D	22	C
3	C	13	D	23	B
4	C	14	A	24	C
5	E	15	D	25	D
6	B	16	E	26	B
7	C	17	C	27	D
8	D	18	C	28	D
9	B	19	A	29	B
10	D	20	C	30	D

Practice Exam 1

1. [**Lesson 2**] $0.07508 = {}_2q_{52} = (d_{52}+d_{53})/l_{52} = 72/l_{52}$, so $l_{52} = 72/0.07508 = 959$. But $l_{52} = l_{50} - d_{50} - d_{51} = 1000 - 20 - d_{51}$, so $d_{51} = \boxed{\mathbf{21}}$. **(B)**

2. [**Section 36.3**] We need to back out q_{59}. We use reserve recursion. Since the insurance is deferred, $1000q_{59}$ is not subtracted from the left side.

$$({}_9V + P)(1.1^{0.5}) = {}_{9.5}V(1-0.5q_{59})$$
$$252(1.1^{0.5}) = 266.70 - 133.35q_{59}$$
$$q_{59} = \frac{2.40017}{133.35} = 0.018$$

Then the benefit reserve at time 10 is, by recursion from time 9,

$$\frac{252(1.1)}{1-0.018} = \boxed{\mathbf{282.28}} \qquad \textbf{(D)}$$

3. [**Lesson 22**] We calculate the net premium per unit.

$$\ddot{a}_{60:\overline{2}|} = 1 + \frac{0.98}{1.05} = 1.93333$$
$$A^{\,1}_{60:\overline{2}|} = \frac{0.02}{1.05} + \frac{(0.98)(0.04)}{1.05^2} = 0.054603$$
$$25.41 = \frac{SA^{\,1}_{60:\overline{2}|}}{\ddot{a}_{60:\overline{2}|}}$$
$$\text{Premium per unit} = \frac{1.93333S}{0.054603} = 0.028243$$

Now we recalculate at 4%.

$$\ddot{a}_{60:\overline{2}|} = 1 + \frac{0.98}{1.04} = 1.94231$$
$$A^{1}_{60:\overline{2}|} = \frac{0.02}{1.04} + \frac{(0.98)(0.04)}{1.04^2} = 0.055473$$
$$\text{Premium per unit} = \frac{1.94231}{0.055473} = 0.028561$$

So the revised premium is 25.41(0.028561/0.028243) = **25.696**. **(C)**

4. [**Lesson 42**]

$$_tp_x^{(\tau)} = \left(\frac{10}{10+t}\right)\left(\frac{10}{10+t}\right)^3 = \left(\frac{10}{10+t}\right)^4$$
$$\mu^{(1)}_{x+t} = -\frac{\mathrm{d}\ln {}_tp_x'^{(1)}}{\mathrm{d}t}$$
$$= -\frac{\mathrm{d}\bigl(\ln 10 - \ln(10+t)\bigr)}{\mathrm{d}t}$$
$$= \frac{1}{10+t}$$
$$q_x^{(1)} = \int_0^1 {}_tp_x^{(\tau)}\mu^{(1)}_{x+t}\,\mathrm{d}t$$
$$= \int_0^1 \left(\frac{10}{10+t}\right)^4\left(\frac{1}{10+t}\right)\mathrm{d}t$$
$$= \int_0^1 \frac{10^4\mathrm{d}t}{(10+t)^5}$$
$$= -\left(\frac{10^4}{4}\right)\left(\frac{1}{(10+t)^4}\right)\Bigg|_0^1$$
$$= \left(\frac{10^4}{4}\right)\left(\frac{1}{10^4} - \frac{1}{11^4}\right)$$
$$= \boxed{\mathbf{0.079247}} \quad \textbf{(C)}$$

5. [**Lesson 10**] The expected present value of one unit of a 10-year deferred whole life insurance is ${}_{10}E_x\,\bar{A}_{x+10}$. The force of mortality is constant after 10 years, so

$$\bar{A}_{x+10} = \frac{\mu}{\mu+\delta} = \frac{0.02}{0.02+0.06} = 0.25$$

The pure endowment factor ${}_{10}E_x$ is computed using the mortality rate in effect for the first ten years, so it is $e^{-(0.01+0.06)(10)} = e^{-0.7}$. Therefore, the EPV of the 10-year deferred whole life insurance is

$$1000\,{}_{10|}\bar{A}_x = 250e^{-0.7}$$

Let P be the single benefit premium. The interest on the benefit in the first ten years at $\delta' = 0.03$ partially offsets the $\delta = 0.06$ discount factor, so the EPV of the first ten years of insurance is

$$\frac{P\mu}{\mu+\delta-\delta'}\left(1-e^{-10(\mu+\delta-\delta')}\right) = \frac{0.01P}{0.04}\left(1-e^{-0.4}\right)$$

We now solve for P.

$$\begin{aligned}P &= 250e^{-0.7} + P\Big(0.25(1-e^{-0.4})\Big)\\ &= 124.1463 + 0.08242P\\ P &= \frac{124.1463}{1-0.0842} = \boxed{\mathbf{135.30}} \qquad \textbf{(E)}\end{aligned}$$

6. [**Lesson 43**] From $d_x^{(1)} = 90$, $q_x^{(1)} = \frac{90}{1000} = 0.09$. Then

$$\begin{aligned}q_x^{(1)} &= q_x'^{(1)}\left(1 - \frac{q_x'^{(2)}}{2}\right)\\ 0.09 &= q_x'^{(1)}\left(1 - q_x'^{(1)}\right)\\ \left(q_x'^{(1)}\right)^2 &- q_x'^{(1)} + 0.09 = 0\\ q_x'^{(1)} &= \frac{1 \pm \sqrt{0.64}}{2} = 0.1, 0.9\end{aligned}$$

The solution 0.9 is rejected since then $q_x'^{(2)} > 1$.

$$q_x'^{(2)} = 0.2 \qquad q_x'^{(3)} = 0.3$$

Since (3) occurs at the end of the year, only $l_x^{(\tau)}\, p_x'^{(1)}\, p_x'^{(2)} = (1000)(0.9)(0.8) = 720$ lives are subject to it. So $d_x^{(3)} = 720(0.3) = \boxed{\mathbf{216}}$. **(B)**

7. [**Lesson 16**] This annuity is the sum of a 20-year certain annuity-due and a 20-year deferred 20-year temporary life annuity due.

$$\begin{aligned}\ddot{a}_{\overline{20|}} &= \frac{1-(1/1.06)^{20}}{1-1/1.06} = 12.15812\\ {}_{20|}\ddot{a}_{30:\overline{20|}} &= {}_{20|}\ddot{a}_{30} - {}_{40|}\ddot{a}_{30}\\ &= {}_{20}E_{30}\,\ddot{a}_{50} - {}_{40}E_{30}\,\ddot{a}_{70}\\ &= (0.29374)(13.2668) - \left(\frac{l_{70}}{l_{30}}\right)\left(\frac{1}{1.06}\right)^{40}(8.5693)\\ &= 3.89699 - \left(\frac{6{,}616{,}155}{9{,}501{,}381}\right)(0.097222)(8.5693)\\ &= 3.89699 - (0.067699)(8.5693)\\ &= 3.89699 - 0.58013 = 3.31686\end{aligned}$$

The expected present value of the annuity is $12.15812 + 3.31686 = \boxed{\mathbf{15.4750}}$. **(C)**

8. [**Lesson 33**] By the insurance ratio formula (33.2),

$$\begin{aligned}A_{30} &= 0.4 + 0.01(30) = 0.7\\ A_{50} &= 0.4 + 0.01(50) = 0.9\\ {}_{20}V &= \frac{A_{50} - A_{30}}{1 - A_{30}} = \frac{0.9-0.7}{1-0.7} = \boxed{\mathbf{\frac{2}{3}}} \qquad \textbf{(D)}\end{aligned}$$

9. [**Lessons 38 and 41**] Since we're just interested in the first transition, this is a multiple-decrement question. We're asked for ${}_{\infty}p_x^{01}$, and by formula (41.2) that is

$$\begin{aligned}
{}_{\infty}p_{40}^{01} &= \int_0^\infty {}_sp_{40}^{00}\mu_{40+s}^{01}\,\mathrm{d}s \\
{}_sp_{40}^{00} &= \exp\left(-\int_0^s \Big(0.002(40+u)+0.0005(40+u)\Big)\mathrm{d}u\right) = e^{-0.00125[(40+s)^2-40^2]} \\
{}_{\infty}p_x^{01} &= \int_0^\infty e^{-0.00125[(40+s)^2-40^2]}\Big(0.002(40+s)\Big)\mathrm{d}s \\
&= -\frac{0.002}{0.0025}e^{0.00125(40^2)}\, e^{-0.00125(40+s)^2}\Big|_0^\infty \\
&= \boxed{\mathbf{0.8}} \qquad \textbf{(B)}
\end{aligned}$$

This could also be done more simply by using the fact that in a multiple decrement model in which the forces are constant proportions of each other, the probability of a specific decrement ever happening is the ratio of its force to the total of the forces. This is discussed on page 824, right after Quiz 41-1.

10. [**Lesson 13**] The benefits are a 9-year decreasing insurance minus twice a 4-year decreasing insurance. $2.8-2(0.7)=\boxed{\mathbf{1.4}}$. **(D)**

11. [**Lesson 5**] Let T be time to breakdown. Let's first calculate $\mathbf{E}[\min(T,5)]$ as the integral of the survival function. For this beta distribution, $S_0(t)=\Big((10-t)/10\Big)^2$.

$$\mathbf{E}[\min(T,5)] = \int_0^5 \left(\frac{10-t}{10}\right)^2 \mathrm{d}t = \frac{1}{300}(10^3-5^3) = 2\tfrac{11}{12}$$

This includes all automobiles. We remove those which last 5 years, which contribute 5 to this expression. $S_0(5)=\left(\frac{1}{2}\right)^2=\frac{1}{4}$, so we remove $5\left(\frac{1}{4}\right)=\frac{5}{4}$ and get $2\frac{11}{12}-\frac{15}{12}=\frac{5}{3}$. Then we divide by the probability of lasting less than 5 years, or $1-S_0(5)=1-\frac{1}{4}=\frac{3}{4}$ to obtain $\left(\frac{5}{3}\right)\left(\frac{4}{3}\right)=\frac{20}{9}=\boxed{\mathbf{2.22222}}$. **(C)**

12. [**Section 26**] The expected loss per policy is

$$\begin{aligned}
\mathbf{E}[L_0] &= \bar{A}_{x:\overline{20}|}\left(S+\frac{P}{\delta}\right)-\frac{P}{\delta} \\
\bar{A}_{x:\overline{20}|} &= \bar{A}^{1}_{x:\overline{20}|}+\bar{A}_{x:\overline{20}|}^{\ \ 1} \\
&= \frac{0.02(1-e^{-0.06(20)})}{0.06}+e^{-0.06(20)} = 0.5341295 \\
\mathbf{E}[L_0] &= 0.5341295\left(1000+\frac{46}{0.04}\right)-\left(\frac{46}{0.04}\right) = -1.62163
\end{aligned}$$

The variance of the loss per policy is, using 2δ

$$\begin{aligned}
{}^2\bar{A}_x &= \frac{0.02(1-e^{-0.10(20)})}{0.02+2(0.04)}+e^{-0.10(20)} = 0.3082682 \\
\mathrm{Var}(L_0) &= (0.3082682-.5341295^2)\left(1000+\frac{46}{0.04}\right)^2 = 106{,}197
\end{aligned}$$

We want n such that $-1.62163n+1.645\sqrt{106{,}197n}=0$.

$$\begin{aligned}
-1.62163\sqrt{n}+1.645\sqrt{106{,}197}&=0\\
\sqrt{n}&=\frac{1.645\sqrt{106{,}197}}{1.62163}=330.58\\
n&=330.58^2=\boxed{\mathbf{109{,}280}}\qquad\textbf{(D)}
\end{aligned}$$

13. [**Lesson 54**] The salary rate at 35 corresponds to $s_{34.5}$. We need:

$$75{,}000\left(\frac{(s_{62}+s_{63}+s_{64})/3}{s_{34.5}}\right)=75{,}000\left(\frac{1.03^{62}+1.03^{63}+1.03^{64}}{3(1.03)^{34.5}}\right)=\boxed{\mathbf{174{,}200}}\qquad\textbf{(D)}$$

14. [**Section 20.2**] By equation (20.10),

$$\mathring{e}_x=e_x+\tfrac{1}{2}-\tfrac{1}{12}\mu_x$$

Force of mortality for (90) is $\mu_{90}=0.0005+0.0008(1.07^{90})=0.353382$. Thus

$$\mathring{e}_{90}=6.647+0.5-\tfrac{1}{12}(0.353382)=\boxed{\mathbf{7.118}}\qquad\textbf{(A)}$$

15. [**Lesson 35**] The two recursions for the benefit reserve from time 8 to time 9 and time 9 to time 10 are, with the common mortality rate denoted by q:

$$\begin{aligned}
(210.10+P)(1.04)-1000q&=232.22(1-q)\\
(232.22+P)(1.04)-1000q&=255.40(1-q)
\end{aligned}$$

We'll solve for q and for P. Subtracting the first equation from the second,

$$\begin{aligned}
22.12(1.04)&=23.18-23.18q\\
q&=-\frac{(22.12)(1.04)-23.18}{23.18}=0.00755824\\
210.10(1.04)+P(1.04)-7.55824&=232.22(1-0.00755824)=230.4648\\
P(1.04)&=230.4648+7.55824-210.10(1.04)=19.5191\\
P&=18.7683\\
q_{x+10}&=1.1(0.00755824)=0.00831406
\end{aligned}$$

Now we calculate ${}_{11}V$.

$${}_{11}V=\frac{(255.40+18.7683)(1.04)-1000(0.00831406)}{1-0.00831406}=\boxed{\mathbf{279.1418}}\qquad\textbf{(D)}$$

16. [**Section 5.2**] By formula (4.2),

$$\begin{aligned}
p_{60}&=\exp\left(-0.001(1.05^{60})\left(\frac{0.05}{\ln 1.05}\right)\right)=0.981040\\
{}_2p_{60}&=\exp\left(-0.001(1.05^{60})\left(\frac{1.05^2-1}{\ln 1.05}\right)\right)=0.961518
\end{aligned}$$

Then $e_{60:\overline{2}|}=0.981040+0.961518=\boxed{\mathbf{1.9426}}$. **(E)**

17. [**Lesson 26**] The variance of future loss for a gross premium of 25 is

$$\begin{aligned}2{,}000{,}000 &= \operatorname{Var}\left(v^{T_x}\right)\left(1000+\frac{25}{0.06}\right)^2\\ &= \operatorname{Var}\left(v^{T_x}\right)(2{,}006{,}944)\end{aligned}$$

If we replace 25 with 20 (for a 20% discount) in the above formula, it becomes

$$\begin{aligned}\operatorname{Var}(L_0) &= \operatorname{Var}\left(v^{T_x}\right)\left(1000+\frac{20}{0.06}\right)^2\\ &= \operatorname{Var}\left(v^{T_x}\right)(1{,}777{,}778)\end{aligned}$$

We see that this is 1,777,778/2,006,944 times the given variance, so the final answer is

$$\operatorname{Var}(L_0)=\frac{1{,}777{,}778}{2{,}006{,}944}(2{,}000{,}000)=\boxed{\mathbf{1{,}771{,}626}}\qquad \textbf{(C)}$$

18. [**Sections 5.1 and 19.1**] The EPV of a continuously increasing continuous annuity is

$$(\bar{I}\bar{a})_x=\int_0^\infty v^t\,{}_tp_x\,\mathrm{d}t$$

and since $v^t=1$, this is $\int_0^\infty t,{}_tp_x\,\mathrm{d}t$. However,

$$\mathbf{E}[T_x^2]=2\int_0^\infty t\,{}_tp_x\,\mathrm{d}t$$

and in our case,

$$\mathbf{E}[T_x^2]=\operatorname{Var}(T_x)+\mathbf{E}[T_x]^2=822+52^2=3526$$

It follows that $(\bar{I}\bar{a})_x=3526/2=\boxed{\mathbf{1763}}$. **(C)**

19. [**Lesson 57**]

I From the row for year 1, with 0 reserves and expenses, we see that $I_t/P_t=0.06$, so the interest rate is 0.06. ✓

II Looking at the line for $t=6$, we see that the reserve per survivor to time $t-1=5$ is 1425. ✓

III First, the profit in year 3 is $800+1000-20+106.8-700-1092.3=94.50$. We deduce survivorship from the bq_{x+t-1} column, and we see that the mortality rates in the first two years are 0.005 and 0.006, so the profit signature component of year'3 is $(0.995)(0.994)(94.50)=93.46$. ✗.

(A)

20. [**Lesson 25**] Let P^g be the gross premium.

$$\begin{aligned}P^g\ddot{a}_{x:\overline{10|}} &= 800\ddot{a}_{x:\overline{10|}}+0.05P^g\ddot{a}_{x:\overline{10|}}+10.50\,{}_{10|}\ddot{a}_x\\ P^g(0.95) &= 800+10.50\left(\frac{{}_{10|}\ddot{a}_x}{\ddot{a}_{x:\overline{10|}}}\right)\end{aligned}$$

But since the net premium is $1000({}_{10|}\ddot{a}_x/\ddot{a}_{x:\overline{10|}})=800$, it follows that $10.5({}_{10|}\ddot{a}_x/\ddot{a}_{x:\overline{10|}})=(10.5/1000)(800)=8.4$.

$$P^g=\frac{808.4}{0.95}=\boxed{\mathbf{850.9474}}\qquad \textbf{(C)}$$

21. [**Lesson 12**] The present value of the benefit decreases with increasing survival time, so the 95th percentile of the present value of the insurance corresponds to the 5th percentile of survival time. The survival probability is

$$\begin{aligned} {}_tp_{45} &= \exp\left(-\int_0^t 0.001(1.05^{45+u})\,\mathrm{d}u\right) \\ -\ln {}_tp_{45} &= \frac{0.001(1.05^{45+u})}{\ln 1.05}\bigg|_0^t \\ &= \frac{0.001(1.05^{45+t}-1.05^{45})}{\ln 1.05} \end{aligned}$$

Setting ${}_tp_{45} = 0.95$,

$$\begin{aligned} \frac{0.001(1.05^{45+t}-1.02^{45})}{\ln 1.05} &= -\ln 0.95 \\ 1.05^{45+t} &= (-1000\ln 0.95)(\ln 1.05) + 1.05^{45} = 11.48762 \\ 1.05^t &= \frac{11.48762}{1.05^{45}} = 1.27853 \\ t &= \frac{\ln 1.27853}{\ln 1.05} = 5.0361 \end{aligned}$$

The value of Z if death occurs at $t = 5.0361$ is $be^{-5.0361(0.06)}$, so the largest face amount is $1{,}000{,}000e^{5.0361(0.06)} =$ **1,352,786**. **(A)**

22. [**Lesson 49**] For $(x) = 30$ and $(y) = 20$, $\mu_x(t) = \mu_y(t) = 1/(70-t)$, so $\mu_{xy}(t) = 2/(70-t)$, which is a beta with $\omega - x = 70$ and $\alpha = 2$. If you remembered formula (5.9), you could write down the answer immediately:

$$\mathrm{Var}(T) = \frac{\alpha(\omega-x)^2}{(\alpha+1)^2(\alpha+2)} = \frac{2(70^2)}{(3^2)(4)} = \boxed{\mathbf{272.22}} \qquad \textbf{(C)}$$

Otherwise, from first principles: the mean is

$$\mathbf{E}[T] = \frac{\omega - x}{\alpha - 1} = \frac{70}{3}$$

The survival function of the joint status is ${}_tp_{30:20} = \big((70-t)/70\big)^2$.

$$\mathbf{E}[T^2] = 2\int_0^{70} t\left(\frac{70-t}{70}\right)^2 \mathrm{d}t$$

Although I usually avoid integration by parts when an alternative is available, integration by parts works out nicely here and will be used.

$$\begin{aligned} \mathbf{E}[T^2] &= 2\left(-t\,\frac{(70-t)^3}{3(70^2)}\bigg|_0^{70} + \int_0^{70}\frac{(70-t)^3}{3(70^2)}\,\mathrm{d}t\right) \\ &= 2\left(-\frac{(70-t)^4}{12(70^2)}\bigg|_0^{70}\right) = \frac{70^2}{6} \end{aligned}$$

$$\mathrm{Var}(T) = \frac{70^2}{6} - \left(\frac{70}{3}\right)^2 = \frac{70^2}{18} = \boxed{\mathbf{272.22}} \qquad \textbf{(C)}$$

23. [**Lesson 23**] Let P be the annual benefit premium.

$$\begin{aligned}
A_{45:\overline{20|}} &= 1 - d\,\ddot{a}_{45:\overline{20|}} \\
&= 1 - \left(\tfrac{1}{21}\right)(12.6) = 0.4 \\
A^{\;1}_{45:\overline{20|}} &= A_{45:\overline{20|}} - {}_{20}E_{45} = 0.4 - 0.3 = 0.1 \\
P &= \frac{500A^{\;1}_{45:\overline{20|}} + 500A_{45}}{\ddot{a}_{45:\overline{20|}}} \\
&= \frac{500(0.4)+500(0.1)}{12.6} = \boxed{\textbf{11.9048}} \qquad \textbf{(B)}
\end{aligned}$$

24. [**Lesson 52**]

$$\begin{aligned}
\bar{A}_{\overline{xy}} &= \bar{A}_x + \bar{A}_y - \bar{A}_{xy} \\
&= \frac{0.02}{0.07} + \frac{0.03}{0.08} - \frac{0.05}{0.10} = 0.16071 \\
\bar{a}_{xy} &= \frac{1}{0.02+0.03+0.05} = 10
\end{aligned}$$

The annual benefit premium is $1000(0.16071/10) = \boxed{\textbf{16.071}}$. **(C)**

25. [**Lesson 40**] We have ${}_{1|2}q_x^{(1)} = {}_3q_x^{(1)} - q_x^{(1)} = 0.017985 - 0.003 = 0.014985$. Also, $p_x^{(\tau)} = 1 - 0.003 - 0.20 = 0.797$. We set up an equation for ${}_{1|2}q_x^{(1)}$ and solve.

$$\begin{aligned}
{}_{1|}q_x^{(1)} + {}_{2|}q_x^{(1)} &= {}_{1|2}q_x^{(1)} \\
(0.797)(a) + (0.797)(1-a-0.15)(2a) &= 0.014985 \\
0.797a + 1.3549a - 1.594a^2 &= 0.014985 \\
1.594a^2 - 2.1519a + 0.014985 &= 0 \\
a = \frac{2.1519 - \sqrt{4.535129}}{3.188} &= \boxed{\textbf{0.007}} \qquad \textbf{(D)}
\end{aligned}$$

The other solution to the quadratic is rejected since it is greater than 1.

26. [**Lesson 50**] ${}_{20}q^{\;1}_{50:60} - {}_{20}q^{\;\;2}_{50:60} = {}_{20}q_{50}\,{}_{20}p_{60}$, and

$$\begin{aligned}
{}_{20}q_{50} &= 1 - \exp\left(-\int_0^{20} 0.002t\,\mathrm{d}t\right) \\
&= 1 - e^{-0.001(20)^2} = 1 - 0.670320 = 0.329680 \\
{}_{20}p_{60} &= \exp\left(-\int_0^{20} 0.003t\,\mathrm{d}t\right) \\
&= e^{-0.0015(20)^2} = 0.548812 \\
{}_{20}q_{50}\,{}_{20}p_{60} &= (0.329680)(0.548812) = \boxed{\textbf{0.180932}} \qquad \textbf{(B)}
\end{aligned}$$

27. [**Lesson 3**] ${}_{5|}q_{20} = \big(S_0(25) - S_0(26)\big)/S_0(20)$, so we will calculate these three values of $S_0(x)$. (Equivalently, one could calculate ${}_5p_{20}$ and ${}_6p_{20}$ and take the difference.) The integral of μ_x is

$$\int_0^x \mu_u \, \mathrm{d}u = \left(\frac{0.002u^2}{2} + 0.005u\right)\Bigg|_0^x = 0.001x^2 + 0.005x$$

so

$$S_0(20) = \exp\Big(-\big(0.001(20^2) + 0.005(20)\big)\Big) = \exp(-0.5) = 0.606531$$
$$S_0(25) = \exp\Big(-\big(0.001(25^2) + 0.005(25)\big)\Big) = \exp(-0.75) = 0.472367$$
$$S_0(26) = \exp\Big(-\big(0.001(26^2) + 0.005(26)\big)\Big) = \exp(-0.806) = 0.446641$$

and the answer is

$${}_{5|}q_{20} = \frac{0.472367 - 0.446641}{0.606531} = \boxed{\mathbf{0.042415}} \qquad \textbf{(D)}$$

28. [**Lesson 25**] By the equivalence principle,

$$P^g(0.9\ddot{a}_{45:\overline{30|}} - 0.3) = 100{,}100A_{45} + ra_{45} + 200 \qquad (*)$$
$$1000A_{45} = 1000(1 - d\,\ddot{a}_{45}) = 1000\left(1 - \frac{0.06}{1.06}(14.1121)\right) = 201.2$$
$$a_{45} = 14.1121 - 1 = 13.1121$$
$$0.9\ddot{a}_{45:\overline{30|}} - 0.3 = 0.9(13.1121) - 0.3 = 11.7350$$

Substituting into (*),

$$1777.98(11.7350) = 100.1(210.2) + 13.1121r + 200$$
$$r = \frac{1777.98(11.7350) - 100.1(201.2) - 200}{13.1121} = \boxed{\mathbf{40}} \qquad \textbf{(D)}$$

29. [**Lessons 32 and 35**] Because premiums and benefits are the same as for an insurance on (40) through year 20, ${}_{20}V$ must be the same as for a standard 1000 whole life insurance on (40), or

$${}_{20}V_{40} = 1 - \frac{\ddot{a}_{60}}{\ddot{a}_{40}} = 1 - \frac{11.1454}{14.8166} = 0.247776$$

Then by the equivalence principle, this reserve plus expected future benefit premiums must equal expected future benefits. If we let P be the premium after age 60:

$$2000A_{60} = 247.776 + P\ddot{a}_{60}$$
$$2000(0.36913) = 247.776 + P(11.1454)$$
$$P = \frac{2000(0.36913) - 247.776}{11.1454} = 44.0077$$

Now we roll the reserve forward one year.

$${}_{21}V = \frac{({}_{20}V + P)(1+i) - 2000q_{60}}{1 - q_{60}}$$
$$= \frac{(247.776 + 44.0077)(1.06) - 2000(0.01376)}{1 - 0.01376}$$
$$= \boxed{\mathbf{285.70}} \qquad \textbf{(B)}$$

30. [**Lesson 55**]

$$y_2 = 0.018$$
$$y_{12} = 0.044$$
$$(1 + f(2,12))^{10} = \frac{1.044^{12}}{1.018^2} = 1.617746$$
$$f(2,12) = \sqrt[10]{1.617446} - 1 = \boxed{\mathbf{0.0493}} \qquad \textbf{(D)}$$