Preface

Thank you for choosing ACTEX.

Since Exam MFE was introduced in May 2007, there have been quite a few changes to its syllabus and its learning objectives. To cope with these changes, ACTEX decided to launch a brand new study manual, which adopts a completely different pedagogical approach.

The most significant difference is that this edition is fully self-contained, by which we mean that, with this manual, you do not even have to read the "required" text (Derivatives Markets by Robert L. McDonald). By reading this manual, you should be able to understand the concepts and techniques you need for the exam. Sufficient practice problems are also provided in this manual. As such, there is no need to go through the textbook's end-of-chapter problems, which are either too trivial (simple substitutions) or too computationally intensive (Excel may be required). Note also that the textbook's end-of-chapter problems are not at all similar (in difficulty and in format) to the questions released by the Society of Actuaries (SoA).

We do not want to overwhelm students with verbose explanations. Whenever possible, concepts and techniques are demonstrated with examples and integrated into the practice problems.

Another distinguishing feature of this manual is that it covers the exam materials in a different order than it occurs in Derivatives Markets. There are a few reasons for using an alternative ordering:

1. Some topics are repeated quite a few times in the textbook, making students difficult to fully understand them. For example, "estimation of volatility" is discussed four times in Derivatives Markets (Sections 11.4, 12.5, 18.5, 23.2)! In sharp contrast, our study manual presents this topic fully in one single section (Module 3, Lesson 4.1).

2. The focus of the textbook is somewhat different from what SoA expects from the candidates. According to the SoA, the purpose of the exam is to develop candidates' knowledge of the theoretical basis. Nevertheless, the first half of the textbook is almost entirely devoted to applications. Therefore, we believe that reading the textbook or following the textbook's ordering is not the best use of your precious time.

Perhaps you have passed some SoA exams by memorizing formulas. However, from the released exam questions, you can easily tell that it is difficult, if not impossible, to pass Exam MFE/3F simply by memorizing all formulas in the textbook. In this connection, in this writing, we help you really learn the materials. By having the reasoning skills, you will discover that there is not really much to memorize.

To help you better prepare for the exam, we intentionally write the practice problems and the mock exams in a similar format as the released exam and sample questions. This will enable you to, for example, retrieve information more quickly in the real exam. Further, we have integrated the sample and previous exam questions provided by the SoA into the study manual into our examples, our practice problems, and our mock exams. This seems to be a better way to learn how to solve those questions, and of course, you will need no extra time to review those questions.

Our recommended procedure for use of this study manual is as follows:

1. Read the lessons in order.
2. Immediately after reading a lesson, complete the practice problems we provide for that lesson. Make sure that you understand every single practice problem.
3. After studying all 25 lessons, work on the mock exams.

If you find a possible error in this manual, please let us know at the "Customer Feedback" link on the ACTEX homepage (www.actexmadriver.com). Any confirmed errata will be posted on the ACTEX website under the "Errata & Updates" link.

A Note on Rounding and Using the Normal Table

To achieve the desired accuracy, we recommend that you store values in intermediate steps in your calculator. If you prefer not to, please at least keep six decimal places.

When using the standard normal distribution table, do not interpolate.

- Use the nearest z-value in the table to find the probability. Example: Suppose that you are to find $\Pr(Z < 0.759)$, where Z denotes a standard normal random variable. Because the z-value in the table nearest to 0.759 is 0.76, your answer is $\Pr(Z < 0.76) = 0.7764$.
- Use the nearest probability value in the table to find the z-value. Example: Suppose that you are to find z such that $\Pr(Z < z) = 0.7$. Because the probability value in the table nearest to 0.7 is 0.6985, your answer is 0.52.

Syllabus Reference

Module 0 and Module 1

Our Study Manual	The Required Text
Module 0: Review	
0.1	1.4, 5.2 (p.132 only)
0.2	5.1 – 5.3
0.3	2.2, 9.1
Module 1: Risk-Neutral Valuation in Discrete-time	
Lesson 1: Introduction to Binomial Trees	
1.1.1	10.1 (up to the middle of p.318)
1.1.2	10.1 (from the middle of p.318 to the middle of p.319)
1.1.3	10.1 (from p.320 to the middle of p.321)
Lesson 2: Multiperiod Binomial Trees	
1.2.1	10.2, 10.3
1.2.2	10.4
1.2.3	10.1 (from the middle of p.321), 11.3 (from the middle of p.358 to the middle of p.359)
Lesson 3: Options on Other Assets	
1.3.1	10.5 (up to p.331)
1.3.2	10.5 (p.332, up to the middle), 9.1 (formula 9.4 only)
1.3.3	10.5 (from the middle of p.332 to the middle of p.334)
Lesson 4: Pricing with True Probabilities	
1.4.1	11.2 (up to the middle of p.347)
1.4.2	11.2 (second half of p.347 to p.350)
Lesson 5: State Prices	
1.5.1	Appendix 11.B
1.5.2	Sample questions #27
1.5.3	Appendix 11.B

Module 2

Module 2: Risk-Neutral Valuation in Continuous-time	
Lesson 1: Brownian Motion	
2.1.1	11.3 (up to the beginning of p.353)
2.1.2	20.2 (up to the first two lines of p.651)
2.1.3	20.2 (from the bottom of p.653 to the bottom of p.654)
2.1.4	20.3 (up to the middle of p.656)
Lesson 2: Stochastic Calculus	
2.2.1	Scattered in 20.2 and 20.3
2.2.2	20.6 (excluding multivariate Ito's lemma)
2.2.3	Mainly scattered in 20.2 and 20.3, Example 20.1
2.2.4	20.2 (p.651 to p.653)
Lesson 3: Modeling Stock Price Dynamics	
2.3.1	18.1, 18.2
2.3.2	20.1, 18.3
2.3.3	18.4 (up to the end of p.602)
2.3.4	18.4 (from the bottom of p.602 to formula 18.30)
2.3.5	11.3 (from the middle of p.353 to p.354)
Lesson 4: The Sharpe Ratio and the Black-Scholes Equation	
2.4.1	12.1 (p.379)
2.4.2	20.4, 21.2 (p.682, but we generalize the approach here)
2.4.3	21.2 (from p.683 to the middle of p.688)
2.4.4	20.5, 21.3 (except "the backward equation"), 20.7 (except "finding the lease rate" and "valuing a claim on S^aQ^b")

Module 3

Module 3: The Black-Scholes Formula	
Lesson 1: Introduction to the Black-Scholes Formula	
3.1.1	22.1 (up to the middle of p.706)
3.1.2	22.1 (p.706 "ordinary options and gap options"), 12.1 (up to p.378)
3.1.3	12.2
Lesson 2: Greek Letters and Elasticity	
3.2.1	12.3 (p.382 – 385, p.387 up to middle, p.388 "Greek measures for portfolio"), 13.4 (up to p.426)
3.2.2	12.3 (p.386, middle of p.388)
3.2.3	12.3 (starting from p.389 to the end of the section 12.3)
3.2.4	Appendix 13.B
Lesson 3: Risk Management Technique	
3.3.1	13.2, 13.3 (up to the next-to-last paragraph on p.419)
3.3.2	13.4 (p.427 to p.429)
3.3.3	13.3 (the bottom of p.419 to the end of the section), p.431
3.3.4	13.5 (the bottom of p.433 to the end of the section)
Lesson 4: Estimation of Volatility and Expected Rate of Appreciation	
3.4.1	12.5, 23.1, 11.4, 23.2 (up to the middle of p.746), 18.5
3.4.2	18.6

Module 4

Module 4: Further Topics on Option Pricing	
Lesson 1: Exotic Options I	
4.1.1	14.2
4.1.2	Exercise 14.20
4.1.3	14.3
4.1.4	14.4 (except p.455 and example 14.2)
Lesson 2: Exotic Options II	
4.2.1	14.6
4.2.2	9.2
4.2.3	Exercise 14.21
4.2.4	14.5, 22.1 (p.706 “Ordinary options and gap options”)
Lesson 3: Simulation	
4.3.1	19.2, 19.3
4.3.2	19.4
4.3.3	19.5
Lesson 4: General Properties of Options	
4.4.1	9.3 (from p.299 to the first 2 lines on p.304)
4.4.2	9.3 (p.293 – 294 “European versus American options” and “maximum and minimum option prices”)
4.4.3	9.3 (p.297 “time to expiration”)
Lesson 5: Early Exercise for American Options	
4.5.1	9.3 (p.294 to the third paragraph of p.296), 11.1
4.5.2	9.3 (p.296 “Early exercise for puts”)
4.5.3	14.4 (p.455 and Example 14.2)

Module 5

Module 5: Interest Rate Models	
Lesson 1: Binomial Interest Rate Tree	
5.1.1	(Scattered in Chapter 7)
5.1.2	24.4
5.1.3	24.5 (up to Figure 24.9 on p.806)
Lesson 2: The Black Model	
5.2.1	9.1 (p.286 “Options on bonds”)
5.2.2	12.2 (p.381 “Options on futures”)
5.2.3	24.3
Lesson 3: An Equilibrium Equation for Interest Rate Derivatives	
5.3.1	24.1 (p.781 “An equilibrium equations for bonds up to (24.17)”)
5.3.2	24.1 (p.783 the first 2 paragraphs)
5.3.3	24.1 (p.783 to the middle of p.784)
5.3.4	24.1 (the middle of p.783 to the end of section 24.1)
Lesson 4: The Rendleman-Bartter, Vasicek and Cox-Ingersoll-Ross Model	
5.4.1	24.2 (p.785 “The Rendleman-Bartter model)
5.4.2	24.2 (p.786 “The Vasicek model”)
5.4.3	24.2 (p.787 “The Cox-Ingersoll-Ross model” and “Comparing Vasicek and CIR”)

Lesson 2 Exotic Options II

OBJECTIVES

1. To understand the payoff characteristics and be able to price the following exotic options: exchange option, forward start options, and gap options
2. To understand the parity relations of exchange options and gap options

In this lesson we continue with our discussion on exotic options. You need to know how to price the exotic options to be introduced in this lesson apart from knowing their payoff characteristics. However, since the pricing formulas can be derived very easily, you should try to understand the principles illustrated in the proofs.

4. 2. 1 Exchange Options

Exchange option is an option with a payoff that is dependent on two risky assets.

Payoff characteristic

Consider a European exchange option to give up K units of an asset worth $Q(T)$ at time T and receive in return an asset worth $S(T)$. The payoff from the option is

$$[S(T) - KQ(T)]_+.$$

We call S the **underlying** asset and Q the **strike** asset. Define

- $c[S(t), Q(t), t; K, T]$: the time-t price of an option with a time-T payoff of $[S(T) - KQ(T)]_+$, when the time-t prices of the underlying and strike assets are $S(t)$ and $Q(t)$.
- $p[S(t), Q(t), t; K, T]$: the time-t price of an option with a time-T payoff of $[KQ(T) - S(T)]_+$, when the time-t prices of the underlying and strike assets are $S(t)$ and $Q(t)$.

If $Q(t) = 1$ for any t, then c and p are just ordinary call and put options.

Put-call duality for exchange options

We can view the exchange option in either of the following two ways:

(1) An option to **buy** one share of S (which is worth $S(T)$ at time T) by paying $KQ(T)$: we exercise when $S(T) > KQ(T)$ and hence this is a **call** option on S with strike KQ.

(2) K units of an option to **sell** 1 shares of Q (which is worth $Q(T)$ at time T) upon receiving $S(T) / K$: we exercise when $Q(T) < S(T) / K$ and hence this is K units of a **put** option on Q with strike S / K.

Thus, calls and puts are equivalent: a call on S is equivalent to a put on Q. There is not a clear distinction between calls and puts. To make this equivalence clearer, notice that at time T,

$$c[S(T), Q(T), T; K, T] = [S(T) - KQ(T)]_+ = K[S(T)/K - Q(T)]_+ = Kp[Q(T), S(T)\ , T; 1/K, T],$$

we have the put-call duality

$$c[S(t), Q(t), t; K, T] = Kp[Q(t), S(t), t; 1/K, T].$$

In particular,

$$c[S(t), Q(t), t; 1, T] = p[Q(t), S(t), t; 1, T].$$

Similarly, we have

$$p[S(t), Q(t), t; K, T] = Kc[Q(t), S(t), t; 1/K, T].$$

The generalized put-call parity: put-call parity for exchange options

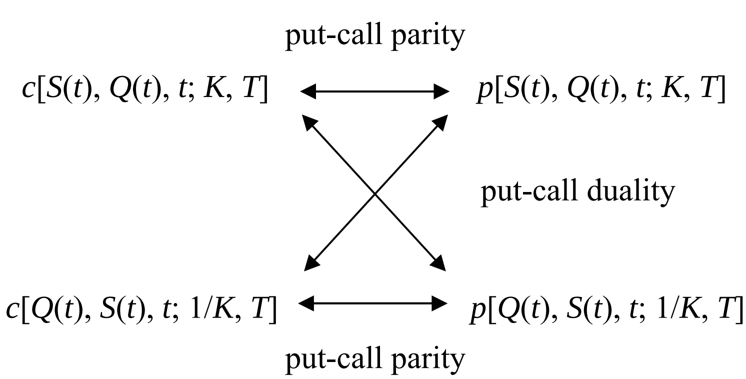
Putting $x = S(T) - KQ(T)$ in the identity $x_+ - x_- = x$, we have

$$[S(T) - KQ(T)]_+ - [KQ(T) - S(T)]_+ = S(T) - KQ(T).$$

The above means that

$$c[S(t), Q(t), t; K, T] - p[S(t), Q(t), t; K, T] = F_{t,T}^P(S) - KF_{t,T}^P(Q)\ .$$

Together with put-call duality, we have the following relations:



Example 4.2.1

You are given that:

(i) S is a nondividend stock whose current price is 20.

(ii) Q is a stock whose current price is 10.

(iii) In the coming 6 months, Q will pay a single dividend of amount 0.8 after 3 months.

(iv) The continuously compounded risk-free rate of interest is 10%.

(v) An option to exchange 2 shares of Q for 1 share of S after 6 months is worth 4.4731.

Calculate the price of an option to exchange 10 share of S for 20 shares of Q after 6 months.

Solution

The prepaid forward price of S is 20. The time-0 prepaid forward price for the delivery of 1 share of Q after 6 months is $10 - 0.8e^{-0.1\times 0.25} = 9.2198$.

By generalized put-call parity,

$$c[S(0), Q(0), 0; 2, 0.5] - p[S(0), Q(0), 0; 2, 0.5] = 20 - 2 \times 9.2198$$

$$p[S(0), Q(0), 0; 2, 0.5] = 4.4731 - 20 + 2 \times 9.2198 = 2.9127.$$

The price of an option to exchange 10 shares of S for 20 shares of Q is 29.127.

[END]

Pricing

Since the payoff of an exchange option depends on two assets, we need a bivariate Black-Scholes model, which is out of the exam MFE syllabus, for the pricing of an exchange option. In Exam MFE, you only need to know how to apply the pricing formula to obtain prices of exchange options.

Let the volatility of S and Q be σ_S and σ_Q, respectively, and ρ be the constant correlation between the continuously compounded returns on S and Q. Then the price of a European option to obtain one unit of S in exchange for K units of Q is

$$F_{t,T}^{P}(S)N(d_1) - KF_{t,T}^{P}(Q)N(d_2)$$

where $d_1 = \dfrac{\ln[F_{t,T}^{P}(S)/KF_{t,T}^{P}(Q)] + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$,

and $\sigma^2 = \sigma_S^2 - 2\rho\sigma_S\sigma_Q + \sigma_Q^2$.

You should notice the similarity of the above formula and the prepaid forward version of the Black-Scholes formula. The only thing that you need to remember is the formula for σ.

Example **4.2.2**

Suppose that for stock S, $S(0) = 20$, $\sigma_S = 30\%$, $\delta_S = 0\%$, for stock Q, $Q(0) = 35$, $\sigma_Q = 50\%$, $\delta_Q = 4\%$. The correlation between the continuously compounded returns on S and Q is $\rho = 0.5$ and the continuously compounded risk-free interest rate is $r = 8\%$.

What is the price of an exchange option with payoff $[3S(1) - 2Q(1)]_+$?

Solution

The payoff can be written as

$$[3S(1) - 2Q(1)]_+ = 3[S(1) - 2Q(1)/3]_+.$$

We price the exchange call with payoff $[S(1) - 2Q(1)/3]_+$.

$$\sigma^2 = 0.3^2 - 2(0.5)(0.3)(0.5) + 0.5^2 = 0.19,\ d_1 = \frac{\ln \frac{20}{35e^{-0.04} \times 2/3} + \frac{0.19}{2}}{\sqrt{0.19}} = -0.0439,$$

$d_2 = -0.0439 - \sqrt{0.19} = -0.47982$, $N(d_1) = 0.4840$, $N(d_2) = 0.3156$.

The price is $20 \times 0.4840 - \frac{2}{3} \times 35e^{-0.04} \times 0.3156 = 2.6047$. The price of 3 units of this option is 7.8142.

[END]

Application to maximum and minimum

The reason to introduce exchange options is that we can use them to price options on the maximum and minimum of two assets.

For maximum,

$$\max[S(T), KQ(T)] = KQ(T) + [S(T) - KQ(T)]_+ = S(T) + [KQ(T) - S(T)]_+$$

and hence

$$V(S(t), Q(t), t) = KF_{t,T}^P(Q) + c[S(t), Q(t), t; K, T] = F_{t,T}^P(S) + p[S(t), Q(t), t; K, T].$$

To price an option with a payoff of $\max[S(T), K]$, we can set $Q(t) = 1$ for all t (and hence $F_{t,T}^P(Q) = e^{-r(T-t)}$), so that

$$V(S(t), 1, t) = Ke^{-r(T-t)} + c(S(t), t; K, T) = F_{t,T}^P(S) + p(S(t), t; K, T).$$

For minimum,

$$\min[S(T), KQ(T)] = S(T) - [S(T) - KQ(T)]_+ = KQ(T) - [KQ(T) - S(T)]_+$$

and hence

$$V(S(t), Q(t), t) = F_{t,T}^P(S) - c[S(t), Q(t), t; K, T] = KF_{t,T}^P(Q) - p[S(t), Q(t), t; K, T].$$

To price an option with a payoff of $\min[S(T), K]$, we can set $Q(t) = 1$ for all t, so that

$$V(S(t), 1, t) = F_{t,T}^{P}(S) - c(S(t), t; K, T) = Ke^{-r(T-t)} - p(S(t), t; K, T).$$

There is a parity relation between options on maximum and minimum. Since

$$\max(a, b) + \min(a, b) = a + b,$$

we have

$$\text{time-}t \text{ price of } \max[S(T), KQ(T)] + \text{time-}t \text{ price of } \min[S(T), KQ(T)]$$
$$= F_{t,T}^{P}(S) + KF_{t,T}^{P}(Q).$$

Example **4.2.3** [MFE May 07 #6]

Consider a model with two stocks. Each stock pays dividends continuously at a rate proportional to its price. $S_j(t)$ denotes the price of one share of Stock j at time t. Consider a claim maturing at time 3. The payoff of the claim is $\max[S_1(3), S_2(3)]$.

You are given:

(i) $S_1(0) = 100$

(ii) $S_2(0) = 200$

(iii) Stock 1 pays dividends of amount $0.05S_1(t)\mathrm{d}t$ between time t and time $t + \mathrm{d}t$.

(iv) Stock 2 pays dividends of amount $0.1S_2(t)\mathrm{d}t$ between time t and time $t + \mathrm{d}t$.

(v) The price of a European option to exchange Stock 2 for Stock 1 at time 3 is $10.

Calculate the price of the claim.

(A) 96 (B) 145 (C) 158 (D) 200 (E) 234

Solution

Statement (iii) and (iv) means that the continuous dividend yields of S_1 and S_2 are 5% and 10%. Statement (v) says that $c(S_1(0), S_2(0), 0; 1, 3) = 10$. The claim is

$$\max[S_1(3), S_2(3)] = S_2(3) + [S_1(3) - S_2(3)]_+.$$

The price of the claim is thus

$$S_2(0)e^{-0.1 \times 3} + c(S_1(0), S_2(0), (0); 1, 3) = 200e^{-0.3} + 10 = 158.16\,.$$

So, the answer is (C).

[END]

4. 2. 2 Currency Options as Exchange Options

Currency option is an excellent example to further illustrate the idea of exchange options. Consider the currencies US dollar ($) and Japanese yen (¥). By c_{US} (c_J) we mean currency options using dollar (yen) as the underlying asset.

– The exchange rate $x_{US}(t)$ $/¥ is the value of ¥1 in terms of US dollars at time t:

$$¥1 \text{ at time } t = \$x_{US}(t)$$

– The exchange rate $y_J(t) = 1/x_{US}(t)$ ¥/$ is the value of $1 in terms of Japanese yen at time t:

$$\$1 \text{ at time } t = ¥y_J(t) = ¥1/x_{US}(t)$$

Consider a currency option that gives the owner the right to buy ¥1 at time T by paying $$K$. The payoff of the option is

$$\max(0, ¥1 - \$K).$$

There are two ways to look at the payoff at time T.

(1) In terms of US dollar, the payoff is

$$(¥1 - \$K)_+ = \$[x_{US}(T) - K]_+.$$

Thus the option is a dollar-yen exchange rate K-strike call, denominated in US dollars. We call this a *dollar-denominated call*. The time-0 price of this call in USD is

$$\$c_{US}(x_{US}(0), K, T).$$

(2) In terms of Japanese yen, the payoff is

$$(¥1 - \$K)_+ = ¥\,[1 - Ky_J(T)]_+ = ¥K\left(\frac{1}{K} - y_J(T)\right)_+.$$

Thus the option is equivalent to K units of yen-dollar exchange rate $1/K$-strike put, denominated in Japanese yen. We call this a *yen-denominated put*. The time-0 price of these K units of put in yen is

$$K \times ¥\, p_J\left(y_J(0), \frac{1}{K}, T\right) = K \times ¥\, p_J\left(\frac{1}{x_{US}(0)}, \frac{1}{K}, T\right).$$

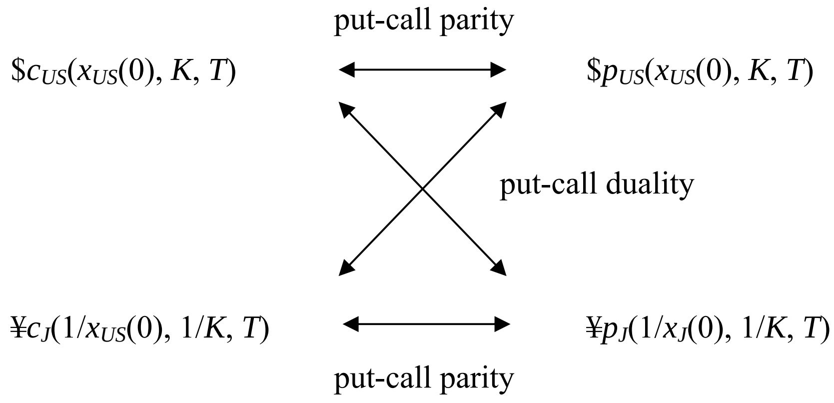
Since $\$c_{US}(x_{US}(0), K, T) = K \times ¥\, p_J\left(\frac{1}{x_{US}(0)}, \frac{1}{K}, T\right)$, we have the following put-call duality:

$$c_{US}(x_{US}(0), K, T) = x_{US}(0)Kp_J\left(\frac{1}{x_{US}(0)}, \frac{1}{K}, T\right).$$

Similarly, we also have

$$p_{US}(x_{US}(0), K, T) = x_{US}(0)Kc_J\left(\frac{1}{x_{US}(0)}, \frac{1}{K}, T\right).$$

Notice that apart from the flip of K to $1/K$ and the adjustment of K at the front, there is a further flip of x_{US} to y_J, and a resulting extra adjustment of $x_{US}(0)$ at the front. To summarize, we have



Example 4.2.4

Suppose the dollar-denominated interest rate is 5%, the yen-denominated interest rate is 1% (both continuously compounded), the spot exchange rate is 0.009\$/¥, and the price of a 1-year dollar-denominated European call to buy one yen with a strike price of \$0.009 is \$0.0006.

What is the no-arbitrage price of a 1-year 1000/9-strike yen-denominated at-the-money call – an option giving the right to buy one dollar, denominated in yen – in Tokyo?

Solution

We are given that

$$c_{US}(0.009, 0.009, 1) = \$0.0006,$$

and we are asked to compute $c_J(1/0.009, 1/0.009, 1)$.

There are two routes to find c_J.

Route 1: use $p_J(1/0.009, 1/0.009, 1)$ as middleman

By put-call duality,

$$p_J(1/0.009, 1/0.009, 1) = (1/0.009)^2 c_{US}(0.009, 0.009, 1) = 0.0006/0.009^2 = 7.4074$$

Then by put-call parity,

$$c_J(1/0.009, 1/0.009, 1) - p_J(1/0.009, 1/0.009, 1) = y_J(0)e^{-r} - \frac{1}{0.009}e^{-r_f}$$

$$c_J(1/0.009, 1/0.009, 1) = 7.4074 + \frac{1}{0.009}e^{-0.05} - \frac{1}{0.009}e^{-0.01} = ¥3.0940$$

Route 2: use $p_{US}(0.009, 0.009, 1)$ as middleman

By put-call parity,

$$c_{US}(0.009, 0.009, 1) - p_{US}(0.009, 0.009, 1) = x_{US}(0)e^{-r_f} - 0.009e^{-r}$$

$$p_{US}(0.009, 0.009, 1) = c_{US}(0.009, 0.009, 1) - 0.009e^{-0.01} + 0.009e^{-0.05} = 2.50616 \times 10^{-4}$$

By put-call duality,

$$c_J(1/0.009, 1/0.009, 1) = (1/0.009)^2 p_{US}(0.009, 0.009, 1) = 2.50616 \times 10^{-4} / 0.009^2 = ¥3.094$$

[END]

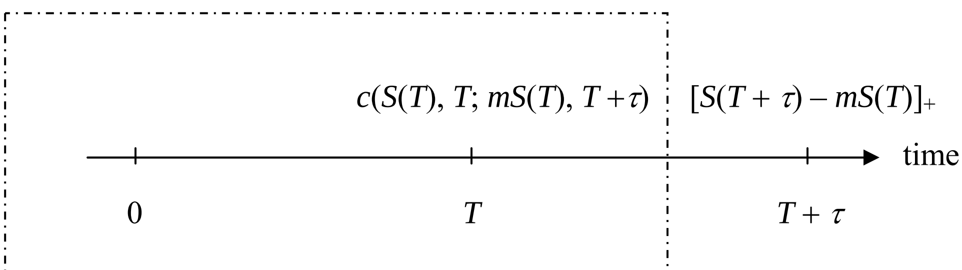
4. 2. 3 Forward Start Options

Forward start option is an exotic option that is not discussed in the main text but in a chapter-end exercise in Chapter 14 of Derivatives Markets.

Payoff characteristic

A forward start call option is a derivative that has a time-T payoff equal to a τ-year call option with a strike that is equal to a multiple of $S(T)$:

$$V(S(T), T) = c(S(T), T; mS(T), T + \tau)$$



A forward start put option can be defined analogously.

Pricing

A forward start option can be priced easily under the Black-Scholes framework. While the time-T payoff looks complicated, actually it can be simplified by using the Black-Sholes formula:

$$\begin{aligned} c(S(T), T; mS(T), T + \tau) &= S(T)e^{-\delta\tau} N(d_1) - mS(T)e^{-r\tau} N(d_2) \\ &= S(T)[e^{-\delta\tau} N(d_1) - me^{-r\tau} N(d_2)], \end{aligned}$$

where

$$d_1 = \frac{\ln\frac{S(T)}{mS(T)} + (r - \delta + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = \frac{-\ln m + (r - \delta + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \text{ and } d_2 = d_1 - \sigma\sqrt{\tau}.$$

Notice that d_1 and d_2 are numerical constants: they do not dependent on S. As a result, the time-T payoff is of the form

$$V(S(T), T) = S(T)\,[e^{-\delta\tau} N(d_1) - me^{-r\tau}N(d_2)],$$

and the time-t price is $F_{t,T}^{P}(S)\,[e^{-\delta\tau} N(d_1) - me^{-r\tau}N(d_2)] = S(t)e^{-\delta(T-t)}[e^{-\delta\tau} N(d_1) - me^{-r\tau}N(d_2)]$.

Example 4.2.5

Assume the Black-Scholes framework for a nondividend-paying stock. You are given that:

(i) The current stock price is 100.

(ii) The stock's volatility is 30%.

(iii) The continuously compounded risk-free interest rate is 5%.

You buy a forward start put option which gives you 6 months from today a 3-month at-the-money put option on the stock. Calculate the premium of the forward start put option.

Solution

We first compute the payoff of the forward start put option:

$$d_1 = \frac{\ln\frac{S(0.5)}{S(0.5)} + (0.05 + \frac{0.3^2}{2}) \times 0.25}{0.3\sqrt{0.25}} = 0.1583,\ d_2 = 0.1583 - 0.3\sqrt{0.25} = 0.0083,$$

$N(-d_1) = 0.4364$, $N(-d_2) = 0.4960$,

$$\begin{aligned} V(S(0.5), 0.5) &= p(S(0.5), 0.5; S(0.5), 0.75) \\ &= S(0.5) \times e^{-0.05\times0.25} \times 0.4960 - S(0.5) \times 0.4364 \\ &= 0.05344S(0.5) \end{aligned}$$

The time-0 price of the option is $0.05344S(0) = 5.344$.

[END]

The textbook gives an application of forward start option that looks very horrible at first glance. It is called a rolling-insurance strategy, which is actually a series of forward start options.

Example **4.2.6**

Assume the Black-Scholes framework for a stock. You are given that:

(i) The current stock price is 100.

(ii) The stock's volatility is 30%.

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(iv) The continuously compounded risk-free interest rate is 8%.

A rolling insurance strategy entails buying one 1-month put option each month, with the strike in each case being 95% of the then-current stock price. What is the time-0 cost of the rolling insurance strategy?

Solution

The option has 12 payoffs at time $T = i / 12$ for $i = 1, 2, \ldots, 12$. The time-T payoff is

$$p(S(T), T; 0.95S(T), T + 1/12).$$

$$d_1 = \frac{\ln \frac{S(T)}{0.95S(T)} + (0.08 - 0.03 + \frac{0.3^2}{2}) \times \frac{1}{12}}{0.3\sqrt{1/12}} = 0.6837,\ d_2 = 0.6837 - 0.3\sqrt{\frac{1}{12}} = 0.5971,$$

$N(-d_1) = 0.2483$, $N(-d_2) = 0.2743$,

$$\begin{aligned} V(S(T), T) &= p(S(T), T; 0.95S(T), T + 1/12) \\ &= 0.95S(T) \times e^{-0.08/12} \times 0.2743 - S(T) \times e^{-0.03/12} \times 0.2483 \\ &= 0.01117S(T) \end{aligned}$$

The time-0 price of the time-T payoff is $0.01117\, F_{0,T}^{P}(S) = 0.01117S(0)e^{-0.03T} = 1.117e^{-0.03T}$.

The time-0 price of the rolling insurance strategy is the sum of the time-0 price of the 12 payoffs:

$$\sum_{i=1}^{12} 1.117e^{-0.03i/12} = 1.117 \times \frac{1 - e^{-0.03}}{e^{0.03/12} - 1} = 13.18844.$$

(The next-to-last step follows from the formula for annuity $\frac{1 - v^n}{i}$ in Exam FM.)

[END]

4. 2. 4 Gap Options

Gap options is a good topic for examination because it can be related to binary options and ordinary call and put options.

Payoff characteristic

Recall that for ordinary call and put options, the payoff functions are

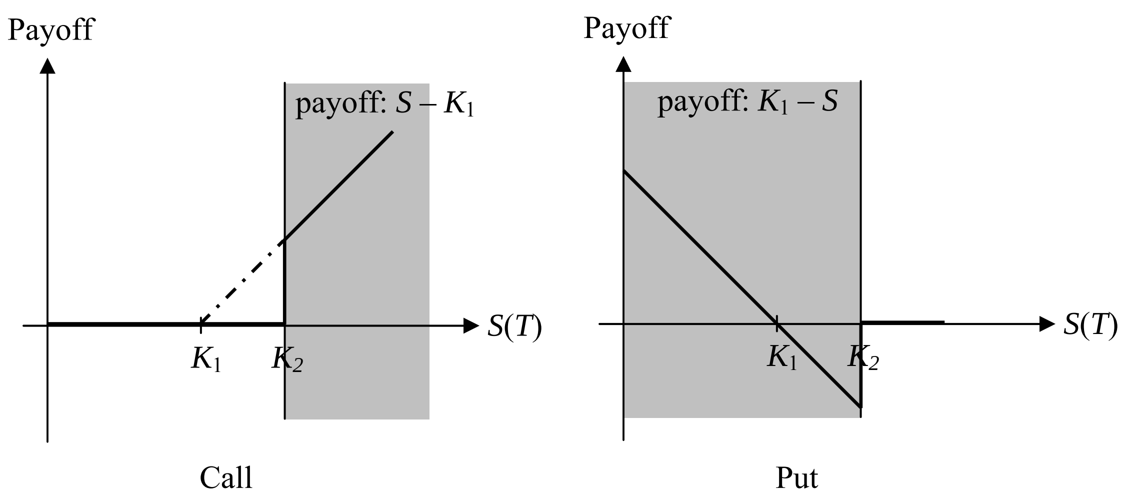
$$\text{Call: } \begin{cases} 0 & \text{when } S(T) \le K \\ S(T) - K & \text{when } S(T) > K \end{cases} \qquad \text{Put: } \begin{cases} K - S(T) & \text{when } S(T) \le K \\ 0 & \text{when } S(T) > K \end{cases}$$

For gap options, we have two Ks:

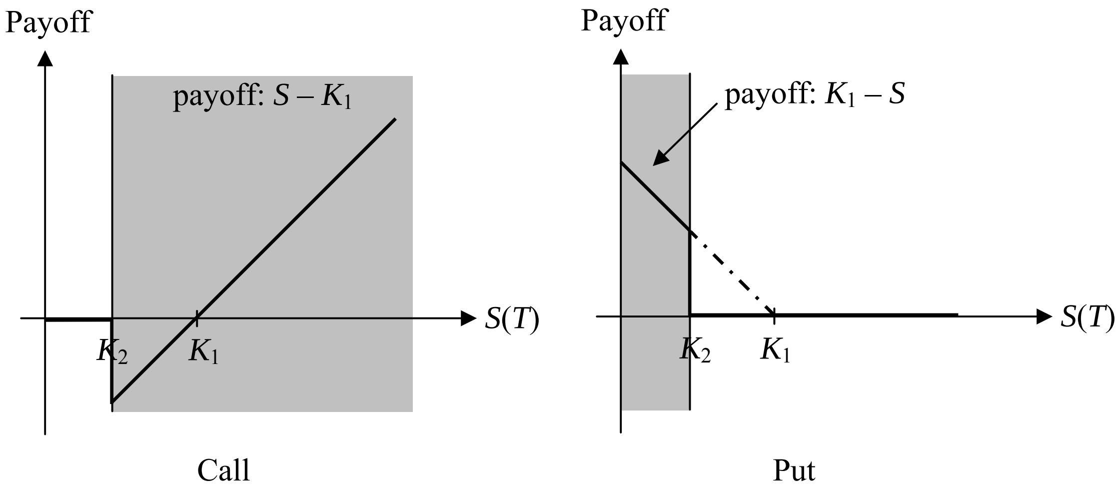
$$\text{Call: } \begin{cases} 0 & \text{when } S(T) \le K_2 \\ S(T) - K_1 & \text{when } S(T) > K_2 \end{cases} \qquad \text{Put: } \begin{cases} K_1 - S(T) & \text{when } S(T) \le K_2 \\ 0 & \text{when } S(T) > K_2 \end{cases}$$

In the above, K_1 is the strike price of the option and K_2 is the payment trigger because it specifies the region where the option would be forced to exercise. The payoff diagrams of the gap options are shown below.

When $K_1 < K_2$:



When $K_1 > K_2$:



Notice that sometimes the option holder is forced to exercise at a loss! Perhaps the name "option" is a misnomer.

Pricing

Under the Black-Scholes framework, gap options can be priced using binary options. For example, for a gap call, the payoff is

$$\begin{aligned} V(S(T), T) &= [S(T) - K_1]\mathrm{I}(S(T) > K_2) \\ &= S(T)\mathrm{I}(S(T) > K_2) - K_1\mathrm{I}(S(T) > K_2), \end{aligned}$$

which shows that a gap call is simply

Gap Call = Asset-or-nothing call (strike K_2) – K_1 × Cash-or-nothing call (strike K_2).

As a result, the time-t price is

$$V(S(t), t) = S(t)e^{-\delta(T-t)}N(d_1) - K_1e^{-r(T-t)}N(d_2),$$

where $d_1 = \dfrac{\ln[S(t)/K_2] + (r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$. It is important to know that the two ds are computed using K_2.

Example 4.2.7

For a stock whose current price is 100, you are given that

(i) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(ii) The stock's volatility is 40%.

(iii) The continuously compounded risk-free rate of interest is 10%.

(a) A call option has a payoff of $S(0.5) - 85$ if $S(0.5) > 105$, and is 0 otherwise. Find the price of the gap call.

(b) A gap put option has a payoff of $80 - S(0.5)$ if $S(0.5) < 105$, and is 0 otherwise.

(c) A gap put option has a payoff of $K - S(0.5)$ if $S(0.5) < 105$, and is 0 otherwise. If the current price of the gap put is 0, find K.

Solution

(a) We first compute d_1 and d_2 using the payment trigger of 105.

$$d_1 = \frac{\ln\frac{100}{105} + (0.1 - 0.03 + \frac{0.4^2}{2}) \times 0.5}{0.4\sqrt{0.5}} = 0.093, \quad d_2 = 0.093 - 0.4\sqrt{0.5} = -0.190$$

so that $N(d_1) = 0.5359$ and $N(d_2) = 0.4247$. The price of the gap call is

$$100e^{-0.03\times0.5} \times 0.5359 - 85e^{-0.1\times0.5} \times 0.4247 = 18.4532.$$

(b) The payoff of the gap put is

$$[80 - S(0.5)]\mathrm{I}(S(0.5) < 105) = 80\mathrm{I}(S(0.5) < 105) - S(0.5)\mathrm{I}(S(0.5) < 105).$$

Thus, a gap put is simply 80 units of cash-or-nothing put minus an asset-or-nothing put. The price of the gap put is

$$80e^{-0.1\times0.5} \times (1 - 0.4247) - 100e^{-0.03\times0.5} \times (1 - 0.5359) = -1.940.$$

Notice that it is possible for the price of gap calls and puts to be negative. In this example, when $80 < S(0.5) < 105$, the payoff of the gap put is negative!

(c) The price of the gap put is

$$Ke^{-0.1\times0.5} \times (1 - 0.4247) - 100e^{-0.03\times0.5} \times (1 - 0.5359) = 0.$$

On solving, we get $K = 83.5444$.

[END]

Gap put-call parity

We know that the prices of ordinary calls and puts are related by put-call parity. For gap calls and puts with the same strike, payment trigger and time to expiration, we have the following gap put-call parity relation:

$$\text{Time-}t \text{ price of gap call} - \text{Time-}t \text{ price of gap put} = F_{t,T}^P(S) - K_1 e^{-r(T-t)}.$$

The reason behind the formula above is simple. Consider the difference of the payoff of the gap call and gap put: If $S(T) > K_2$, then only the gap call would be exercised, and the payoff is

$$S(T) - K_1 - 0.$$

If $S(T) < K_2$, then only the gap put would be exercised, and the payoff is $0 - [K_1 - S(T)]$. In both cases, the payoff is $S(T) - K_1$.

Relation between gap options and ordinary options

Apart from using asset-or-nothing and cash-or-nothing options, we can also use ordinary options and cash-or-nothing options to price gap options. To this end, observe that for a gap call,

$$\begin{aligned} V(S(T), T) &= [S(T) - K_2 + (K_2 - K_1)]\mathrm{I}(S(T) > K_2) \\ &= [S(T) - K_2]\mathrm{I}(S(T) > K_2) + (K_2 - K_1)\mathrm{I}(S(T) > K_2), \end{aligned}$$

which shows that a gap call is simply

$$\text{Ordinary call (strike } K_2) + (K_2 - K_1) \times \text{Cash-or-nothing call (strike } K_2).$$

As a result, the time-t price is

$$V(S(t), t) = c(S(t), t; K_2, T) + (K_2 - K_1)e^{-r(T-t)}N(d_2),$$

where d_2 is calculated based on K_2.

Example **4.2.8** [MFE May 07 #17]

Let $S(t)$ denote the price at time t of a stock that pays dividends continuously at a rate proportional to its price. Consider a European gap option with expiration date T, $T > 0$. If the stock price at time T is greater than \$100, the payoff is $S(T) - 90$; otherwise, the payoff is zero. You are given:

(i) $S(0) = 80$.

(ii) The price of a European call option with expiration date T and strike price 100 is 4.

(iii) The delta of the call option in (ii) is 0.2.

Calculate the price of the gap option.

(A) 3.6 (B) 5.2 (C) 6.4 (D) 10.8 (E) not enough information

Solution

Let d_1 and d_2 be calculated based on the payment trigger $K_2 = 100$. Consider the call option in (ii). We have

$$\Delta = e^{-\delta T}N(d_1) = 0.2,$$

and

$$\text{call price} = 80e^{-\delta T}N(d_1) - 100e^{-rT}N(d_2) = 4,$$

giving $e^{-rT}N(d_2) = 0.12$.

Price of the gap option

= Price of an ordinary call ($K = 100$) + 10 × Price of a cash-or-nothing call ($K = 100$)

$= 4 + 10e^{-rT}N(d_2)$

$= 4 + 10(0.12) = 5.2$

So the answer is (B).

[END]

Greeks of gap options

The formula above can be used to relate the Greeks letters of an ordinary options and gap options. For example, since we know that the delta of 1 unit of cash-or-nothing call is

$$\frac{\partial}{\partial S}e^{-r(T-t)}N(d_2) = e^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S} = e^{-r(T-t)}N'(d_2)\frac{1}{\sigma S\sqrt{T-t}},$$

the time-t delta of a gap call is

$$e^{-\delta(T-t)}N(d_1) + (K_2 - K_1)e^{-r(T-t)}N'(d_2)\frac{1}{\sigma S(t)\sqrt{T-t}}.$$

Exercise 4. 2

Exchange options

1. For two stocks S and Q, you are given:

   (i) The current price of S is 75.

   (ii) The current price of Q is 100.

   (iii) Both stocks pay dividends continuously at a rate proportional to its price. The dividends for S and Q are 3% and 5%, respectively.

   (iv) The current price of an option to exchange 3 units of Q for 4 units of S at time 1 is 86.28.

   Compute the price of an option to exchange 16 units of S for 12 units of Q at time 1.

2. Assume the Black-Scholes framework for a stock whose time-t price is $S(t)$. You are given:

   (i) $S(0) = 75$.

   (ii) S pays dividends of amount $0.03S_1(t)\mathrm{d}t$ between time t and time $t + \mathrm{d}t$.

   (iii) $\mathrm{Var}[\ln S(t)] = 0.09t$.

   (iv) The continuously compounded risk-free interest rate is 0.09.

   Compute the price of the following contingent claims that mature at time 0.5.

   (a) $\max(S(0.5), 80)$;

   (b) $\min(S(0.5), 80)$.

3. You are given that

   (i) S is a nondividend-paying stock that is currently priced at 30.

   (ii) Q is currently priced at 33. It is expected to pay a single dividend after of 1.5 after 2 months in the coming 3-month period.

   (iii) A European derivative that pays the maximum of the stock price of S and Q after 3 months has a current price of 33.329.

   (iv) The continuously compounded risk-free interest rate is 10%.

   (a) Calculate the current price of a derivative that pays the minimum of the stock price of S and Q after 3 months.

   (b) Calculate the current price of a derivative that gives the option holder the right to exchange 1 share of Q for 1 share of S after 3 months.

4. You are given that

   (i) Stock S is a nondividend-paying stock that is currently priced at 30.

(ii) Stock Q is currently priced at 60. It pays dividends at a rate that is proportional to its price. The continuous dividend yield is 4%.

(iii) The volatility of S and Q are 0.3 and 0.5.

(iv) The correlation between the continuously compounded return on S and Q is –0.4.

(v) The continuously compounded risk-free interest rate is 10%.

Calculate the current price of a derivative that gives the option holder the right to exchange 1 share of Q for 2 share of S and Q after 3 months.

5. An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time-t value of one unit of which is denoted by $S(t)$. The contracts offer a minimum guarantee return rate of 4%. At time 0, a single premium of amount π is paid by the policyholder, and $\pi \times y\%$ is deducted by the insurance company. At the contract maturity date, which is 2 years, the insurance company will pay the policyholder

$$\pi \times (1 - y\%) \times \max[S(2)/S(0), 1.04].$$

You are given the following information:

(i) The stock index pays no dividends.

(ii) The index's volatility is 40%.

(iii) The continuously compounded risk-free rate is 10%.

Determine y, so that the insurance company does not make or lose money on this contract.

6. Assume the Black-Scholes framework. You are given that

(i) The current stock price is 10.

(ii) The stock's volatility is 50%.

(iii) The continuously compounded risk-free interest rate is greater than the constant dividend yield of the stock by 5%.

A contingent claim that pays the minimum of S and 10 after 1 year has a current price of 7.628.

Find the dividend yield of the stock.

Currency options as exchange options

7. You are given:

(i) The current yen-HK dollar exchange rate is 12¥/$.

(ii) The HK-denominated continuously compounded risk-free interest rate is 1.5%.

(iii) The Japan-denominated continuously compounded risk-free interest rate is 0.75%.

If the price of a 1-year HK dollar-denominated European call to buy ¥1000 with a strike of $100 is $14.339, find the price of a 1-year yen-denominated European put to sell $200 with a strike of ¥2000.

8. You are given:
   (i) The current euro-US exchange rate is 1.0753€/$.
   (ii) The euro-denominated continuously compounded risk-free interest rate is 4%.
   (iii) The US-denominated continuously compounded risk-free interest rate is 9%.

   If the price of a 6-month dollar-denominated European call to buy one euro with a strike of \$1 is \$0.12037, find the price of a 6-month euro-denominated European call to buy one US dollar with a strike of €1.

9. You are given:
   (i) The current euro-US exchange rate is 1.0526€/$.
   (ii) The euro-denominated continuously compounded risk-free interest rate is 5%.
   (iii) The US-denominated continuously compounded risk-free interest rate is 9%.

   If the price of a 6-month Euro-denominated European put to sell 1 euro with a strike of €0.97 is €0.1528, find the price of a 6-month US-denominated European put to sell one US dollar with a strike of \$1.03093.

10. Suppose that the market price of the call in Example 4.2.4 is \$2.9. Construct an arbitrage opportunity.

Forward start options

11. Assume the Black-Scholes framework for a stock. You are given that:
    (i) The current stock price is 100.
    (ii) The stock's volatility is 20%.
    (iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.
    (iv) The continuously compounded risk-free interest rate is 8%.

    You buy a forward start option which gives you 3 months from today a 6-month at-the-money call option on the stock. Calculate the price of the forward start option.

12. Assume the Black-Scholes framework for a stock. You are given that:
    (i) $\text{Var}[\ln S(0.25)] = 0.0625$.
    (ii) The stock pays no dividends.
    (iii) The continuously compounded risk-free interest rate is 8%.
    (iv) A forward start option that gives the option holder 4 months from now a 3-month at-the-money put option is 5.3363.

    Calculate the price of a 7-month at-the-money European put option.

13. Assume the Black-Scholes framework for a stock. You are given that:

    (i) The stock's volatility is 40%.

    (ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

    (iii) The continuously compounded risk-free interest rate is 10%.

    An investor bought a forward start option that gives you 6 months from now 100 units of 3-month at-the-money call option, with a strike being 110% of the then-current stock price. The investor then immediately delta-hedges his position by trading the underlying stock. How many shares should the investor trade?

14. Assume the Black-Scholes framework for a stock. You are given that:

    (i) The current stock price is 10.

    (ii) The stock's volatility is 30%.

    (iii) The stock is nondividend-paying.

    (iv) The continuously compounded risk-free interest rate is 8%.

    You buy a 3-month call option which gives you the right to buy 100 units of 6-month at-the-money put on the stock at 73. Calculate the price of the call option.

Gap options

15. For a futures index, you are given that

    (i) The time-t value of the index is $F(t)$.

    (ii) $F(0) = 75$.

    (iii) The index's volatility is 35%.

    (iv) The continuously compounded risk-free rate of interest is 10%.

    A European gap call option has a time-1 payoff of $F(1) - K$ if $F(1) > 85$, and is 0 otherwise. If the current price of the gap call is 0, find K.

16. Assume the Black-Scholes framework. Let $S(t)$ denote the price at time t of a nondividend-paying stock. Consider a 6-month European gap option. If the stock price after 6 months is greater than 105, the payoff is $S(0.5) - 90$; otherwise, the payoff is zero.

    You are given:

    (i) $S(0) = 90$.

    (ii) The forward price of a 6-month 105-strike cash-or-nothing put option is 0.7936.

    (iii) The continuously compounded risk-free rate is 5%.

    Calculate the price of the gap option.

17. Let $S(t)$ denote the price at time t of a stock. Consider an 8-month European gap option. If the stock price after 8 months is less than 28, the payoff is $28.5 - S(8/12)$; otherwise, the payoff is zero.

    You are given:

    (i) $S(0) = 30$.

    (ii) The stock will pay a dividend of amount of 2 dollars after 4 months. This is the only dividend that will be paid before the gap option expires.

    (iii) The prepaid forward price of the stock follows a geometric Brownian motion with a volatility of 33%.

    (iv) The continuously compounded risk-free interest rate is 10%.

    Calculate the price of the gap option.

18. Let $S(t)$ denote the price at time t of a stock. You are given that

    (i) $S(0) = 10$.

    (ii) The stock pays no dividends.

    (iii) The stochastic process $S(t)$ is

    $$dS(t) = 0.24S(t)dt + 0.2S(t)dZ(t)$$

    where $Z(t)$ is a standard Brownian motion.

    (iv) The continuously compounded risk-free interest rate is 10%.

    Calculate the expected payoff of a gap option that pays $S(1) - 11$ if and only if $S(1)$ is greater than 10.

19. You are given:

    (i) The current stock price is 40.

    (ii) The stock will pay dividends of 1 dollar each after 3 months and 6 months.

    (iii) The price of a 7-month at-the-money call on the stock is 3.45.

    (iv) The price of a 7-month at-the-money cash-or-nothing put on the stock is 0.53.

    (v) The continuously compounded risk-free interest rate is 10%.

    Consider a 7-month European gap option and let $S(7/12)$ be the stock price after 7 months. If the stock price after 7 months is less than 40, the payoff is $38 - S(7/12)$; otherwise, the payoff is zero.

    Calculate the price of the gap option.

20. Let $S(t)$ be the time-t stock price. You are given:

    (i) $S(0) = 45$.

    (ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 5%.

(iii) The price of a 1-year gap call with a payoff of $S(1) - 46$ if $S(1) > 45$ is 5.633.

(vi) The price of a 1-year at-the-money cash-or-nothing put on the stock is 0.432.

(v) The continuously compounded risk-free interest rate is 10%.

Consider a 1-year European gap option. If the stock price after 1 year is less than or equal to 45, the payoff is $43 - S(1)$; otherwise, the payoff is zero.

Calculate the price of the gap option.

Solutions to Exercise 4.2

1. For the payoff $[4S(1) - 3Q(1)]_+$, the price is 86.28. We are asked to compute the price of the payoff $[12Q(1) - 16S(1)]_+ = 4[3Q(1) - 4S(1)]_+$.

   By the generalized put-call parity,

   $$\text{Price of } [3Q(1) - 4S(1)]_+ - \text{Price of } [4S(1) - 3Q(1)]_+ = 3F^P_{0,1}(Q) - 4F^P_{0,1}(S).$$

   $$\text{Price of } [3Q(1) - 4S(1)]_+ = 86.28 + 3(100e^{-0.05}) - 4(75e^{-0.03}) = 80.51517.$$

   The price of $4[3Q(1) - 4S(1)]_+$ is 322.06.

2. From (ii), $\delta = 0.03$. From (iii), $\sigma^2 = 0.09$ (or $\sigma = 0.3$).

   $$d_1 = \frac{\ln\frac{75}{80} + (0.09 - 0.03 + \frac{0.09}{2}) \times 0.5}{0.3\sqrt{0.5}} = -0.0568,\ d_2 = -0.269.$$

   $N(d_1) = 0.4761$, $N(d_2) = 0.3936$.

   $c(S(0), 0; 80, 0.5) = 75e^{-0.015} \times 0.4761 - 80e^{-0.045} \times 0.3936 = 5.07344.$

   (a) The price is $80e^{-0.045} + 5.07344 = 81.5532$.

   (b) The price is $75e^{-0.015} - 5.07344 = 68.8100$.

3. The time-0 prepaid forward price for the delivery of 1 share of S is 30.

   The time-0 prepaid forward price for the delivery of 1 share of Q is $33 - 1.5e^{-0.1/6} = 31.5248$.

   (a) Since the sum of the prices of options on the maximum and minimum of two assets is the sum of the prepaid forward prices of the two assets,

   Price of the minimum of S and $Q = 30 + 31.5248 - 33.329 = 28.1958$.

   (b) Since $Q(0.25) + [S(0.25) - Q(0.25)]_+ = \max[S(0.25), Q(0.25)]$, the price of the exchange option is $33.329 - 31.5248 = 1.804$.

4. $F^P_{0,0.25}(Q) = 60e^{-0.04\times 0.25} = 60e^{-0.01}$, $\sigma^2 = 0.3^2 - 2(-0.4)(0.3)(0.5) + 0.5^2 = 0.46$.

$$d_1 = \frac{\ln\frac{2\times 30}{60e^{-0.01}} + \frac{0.46}{2}\times 0.25}{\sqrt{0.46\times 0.25}} = 0.1990,\ d_2 = 0.1990 - \sqrt{0.46\times 0.25} = -0.1401.$$

$N(d_1) = 0.5793$, $N(d_1) = 0.4443$.

The price of the exchange option is $60 \times 0.5793 - 60e^{-0.01} \times 0.4443 = 8.3652$.

5. We can treat $Y(t) = S(t)/S(0)$ as a stock with a current price of 1 and volatility of 40%. The time-0 price of $\max[Y(2), 1.04]$ is given by the sum of the prepaid forward price of the strike and a call on Y:

$$1.04e^{-0.2} + c(Y(0), 0; 1.04, 2).$$

To find the value of the call,

$$d_1 = \frac{\ln\frac{1}{1.04} + (0.1 + \frac{0.4^2}{2})\times 2}{0.4\sqrt{2}} = 0.5671,\ d_2 = 0.0014,\ N(d_1) = 0.7157,\ N(d_2) = 0.5.$$

$c(Y(0), 0; 1.04, 2) = 0.7157 - 1.04e^{-0.2} \times 0.5 = 0.28996$.

Thus, the time-0 price of the payment is

$$\pi \times (1 - y\%) \times (1.04e^{-0.2} + 0.28996) = 1.14144\,\pi \times (1 - y\%)$$

Equating the above to π, we get $1.14144\,(1 - y\%) = 1$, or $y = 12.39$.

6. The price of $\min(S(1), 10)$ is

$10e^{-\delta} - c(S(0), 0; 10, 1)$

$= 10^{-\delta} - 10e^{-\delta}N(d_1) + 10e^{-r}N(d_2)$

$= 10e^{-r}N(d_2) + 10e^{-\delta}[1 - N(d_1)]$,

where $d_1 = \dfrac{\ln\frac{10}{10} + r - \delta + \frac{0.5^2}{2}}{0.5} = \dfrac{0 + 0.05 + \frac{0.5^2}{2}}{0.5} = 0.35$, $d_2 = 0.35 - 0.5 = -0.15$,

$N(d_1) = 0.6368$, $N(d_2) = 0.4404$.

Thus,

$$10e^{-(\delta + 0.05)} \times 0.4404 + 10e^{-\delta} \times (1 - 0.6368) = 7.628.$$

On solving, we get $\delta = 0.025$.

7. The payoff of the call is $\$[1000x(1) - 100]_+ = \$1000[x(1) - 0.1]_+$.This is 1000 units of yen call with a strike of 0.1. As a result, the price of a 1-year HK dollar-denominated European call to buy 1 yen with a strike of 0.1 is 0.014339. By put-call duality, the price of a 1-year yen-denominated European put to buy \$1 with a strike of ¥10 is

$$p_J(12, 10, 1) = 120c_{HK}(12^{-1}, 10^{-1}, 1) = 0.014339 \times 120 = ¥1.72068.$$

The price of the put is 200 × ¥1.72068 = ¥344.136.

8. By put-call duality,

$$c_E(1.0753, 1, 0.5) = 1.0753p_{US}(1.0753^{-1}, 1^{-1}, 0.5) = 1.0753p_{US}(0.93, 1, 0.5),$$

where $c_{US}(0.93, 1, 0.5) - p_{US}(0.93, 1, 0.5) = 0.93e^{-0.02} - e^{-0.045}$, which gives

$$p_{US}(0.93, 1, 0.5) = 0.12037 - 0.93e^{-0.02} + e^{-0.045} = 0.16481$$

and hence $c_E(1.0753, 1, 0.5) = 1.0753 \times 0.16481 = 0.17722$.

9. By put-call duality,

$$p_{US}(1.0526^{-1}, 1.03093, 0.5) = 1.0526^{-1}1.03093c_E(1.0526, 1.03093^{-1}, 0.5)$$
$$= 0.979413c_E(1.0526, 0.97, 0.5),$$

where $c_E(1.0526, 0.97, 0.5) - p_E(1.0526, 0.97, 0.5) = 1.0526e^{-0.045} - 0.97e^{-0.025}$, which gives

$$c_E(1.0526, 0.97, 0.5) = 0.1528 + 1.0526e^{-0.045} - 0.97e^{-0.025} = 0.21303,$$

and hence $p_{US}(1.0526^{-1}, 1.3093, 0.5) = 0.979413 \times 2.1303 = 0.20865$.

10. By the principle "buy low, sell high", we buy the yen-denominated call from the market, and then sell a yen-denominated call that is synthesized by (1000/9) units of dollar-denominated put with a strike of 0.009. Since we do not know if this put is selling in the market, we further synthesize the put by using put-call parity:

dollar-denominated put = dollar-denominated call + $0.009\exp(-r_{US}) - 0.009\exp(-r_J)$.

As a result, we sell (1000/9) units of dollar-denominated call, short $\exp(-r_{US})$ in US at the US risk-free rate, and invest $\$\exp(-r_J) = ¥(1000/9)\exp(-r_J)$ in Japan at the Japanese risk-free rate:

Position	Cost (in ¥)	Payoff after 1 year (in ¥)	
		$y_J(1) > 1000/9$	$y_J(1) < 1000/9$
long ¥-denominated \$ call	2.9	¥($y_J(1)$ – 1000/9)	0
long ¥(1000/9)$e^{-0.01}$ in Japan bank	(1000/9)$e^{-0.01}$	¥1000/9	¥1000/9
short \$$e^{-0.05}$ in US bank	–(1000/9)$e^{-0.05}$	–\$1 = –¥$y_J(1)$	–¥$y_J(1)$
short 1000/9 \$-denominated ¥ call	–1000/9 × 0.0006 × 1000/9	0	–\$(1000/9)[$x_{US}(1) - 0.009$] = –¥ [1000/9 – $y_J(1)$]
net position	–0.194029	0	0

11. We first compute the payoff of the forward start call option:

$$d_1 = \frac{\ln\frac{S(0.25)}{S(0.25)} + (0.08 - 0.03 + \frac{0.2^2}{2}) \times 0.5}{0.2\sqrt{0.5}} = 0.2475, \; d_2 = 0.2475 - 0.2\sqrt{0.5} = 0.1061,$$

$N(d_1) = 0.5987$, $N(d_2) = 0.5438$,

$V(S(0.25), 0.25) = c(S(0.25), 0.25; S(0.25), 0.75)$

$= S(0.25) \times e^{-0.03\times 0.5} \times 0.5987 - S(0.25) \times e^{-0.08\times 0.5} \times 0.5438$

$= 0.06731S(0.25)$

The time-0 price of the option is $0.06731S(0)e^{-0.03\times 0.25} = 6.68063$.

12. Since ln $S(t)$ is normally distributed with a variance of $\sigma^2 t$, statement (ii) gives $0.25\sigma^2 = 0.0625$, and hence $\sigma = 0.5$. We then compute the payoff of the forward start put option:

$$d_1 = \frac{\ln\frac{S(1/3)}{S(1/3)} + (0.08 + \frac{0.5^2}{2}) \times 0.25}{0.5\sqrt{0.25}} = 0.205,\ d_2 = 0.205 - 0.5\sqrt{0.25} = -0.045,$$

$N(d_1) = 0.5832$, $N(d_2) = 0.4801$,

$V(S(1/3), 1/3) = p(S(1/3), 1/3; S(1/3), 7/12)$

$= S(1/3) \times e^{-0.08\times 0.25} \times (1 - 0.4801) - S(1/3) \times (1 - 0.5832)$

$= 0.092805S(1/3)$

The time-0 price of the option is $0.092805S(0) = 5.3363$. This gives $S(0) = 57.5$.

Finally, we compute $p(S(0), 0; S(0), 7/12)$:

$$d_1 = \frac{(0.08 + \frac{0.5^2}{2}) \times \frac{7}{12}}{0.5\sqrt{7/12}} = 0.3131,\ d_2 = 0.3131 - 0.5\sqrt{\frac{7}{12}} = -0.0687,$$

$N(d_1) = 0.6217$, $N(d_2) = 0.4721$,

$p(S(0), 0; S(0), 7/12) = 57.5e^{-0.08\times 7/12} \times (1 - 0.4721) - 57.5 \times (1 - 0.6217) = 7.218$.

13. We first compute the payoff of the forward start call option:

$$d_1 = \frac{\ln\frac{S(0.5)}{1.1S(0.5)} + (0.1 - 0.02 + \frac{0.4^2}{2}) \times 0.25}{0.4\sqrt{0.25}} = -0.2766,$$

$$d_2 = -0.2766 - 0.4\sqrt{0.25} = -0.4766,$$

$N(d_1) = 0.3897$, $N(d_2) = 0.3156$,

$V(S(0.5), 0.5) = 100c(S(0.5), 0.5; 1.1S(0.25), 0.75)$

$= 100[S(0.5) \times e^{-0.02\times 0.25} \times 0.3897 - 1.1S(0.5) \times e^{-0.1\times 0.25} \times 0.3156]$

$= 4.916777S(0.5)$

The time-0 price of the option is $4.916777S(0)e^{-0.02\times 0.5} = 4.8679S(0)$.

As a result, the investor should short sell 4.87 shares of the underlying stock.

14. We first compute the payoff of the underlying asset, which is a 6-month at-the-money put $100p(S(0.25), 0.25; S(0.25), 0.75)$:

$$d_1 = \frac{\ln\frac{S(0.25)}{S(0.25)} + (0.08 + \frac{0.3^2}{2}) \times 0.5}{0.3\sqrt{0.5}} = 0.2946,\ d_2 = 0.2946 - 0.3\sqrt{0.5} = 0.0825,$$

$N(-d_1) = 0.3859$, $N(-d_2) = 0.4681$,

$V(S(0.25), 0.25) = [100p(S(0.25), 0.25; S(0.25), 0.75) - 73]_+$

$= 100[S(0.25) \times e^{-0.08\times 0.5} \times 0.4681 - S(0.25) \times 0.3859 - 0.73]_+$

$= 100[0.063846S(0.25) - 0.73]_+$

$= 6.3846[S(0.25) - 11.43385]_+$

which is 6.3846 units of 3-month 11.43385-strike call on S.

The time-0 price is computed using the Black-Scholes formula as below:

$$d_1 = \frac{\ln\frac{S(0)}{11.43385} + (0.08 + \frac{0.3^2}{2}) \times 0.25}{0.3\sqrt{0.25}} = -0.6850,\ d_2 = -0.6850 - 0.3\sqrt{0.25} = -0.8350,$$

$N(d_1) = 0.2483$, $N(d_2) = 0.2033$,

$V(S(0.25), 0.25) = 6.3846[10 \times 0.2483 - 11.43385 \times e^{-0.02} \times 0.2033] = 1.3058.$

15. We have $d_1 = \dfrac{\ln\frac{75}{85} + \frac{0.35^2}{2}}{0.35} = -0.1826$, $d_2 = -0.1826 - 0.35 = -0.5326$, $N(d_1) = 0.4286$, $N(d_2) = 0.2981$.

The time-0 price of the gap call is

$$F(0)N(d_1) - Ke^{-rT}N(d_2) = 75e^{-0.1} \times 0.4286 - Ke^{-0.1} \times 0.2981 = 0.$$

On solving, we get $K = 107.833$.

16. The forward price of the cash-or-nothing put is $N(-d_2) = 1 - N(d_2) = 0.7936$. So, $d_2 = -0.82$.

$$\frac{\ln\frac{90}{105} + (0.05 - \frac{\sigma^2}{2}) \times 0.5}{\sigma\sqrt{0.5}} = -0.82$$

$$0.25\sigma^2 - 0.82\sqrt{0.5}\sigma - \ln\frac{90}{105} - 0.025 = 0$$

$\sigma = 2.07$ (too big, reject) or 0.2496. So, $d_1 = -0.82 + 0.2496\sqrt{0.5} = -0.6435$.

The price of the gap option is

$S(0)N(d_1) - 90e^{-rT}N(d_2) = 90 \times 0.2611 - 90e^{-0.025} \times (1 - 0.7936) = 5.3816.$

17. The prepaid forward price is $F_{0,8/12}^{P}(S) = 30 - 2e^{-0.1\times 4/12} = 28.0656$.

$$d_1 = \frac{\ln\frac{28.0656}{28e^{-0.1\times 8/12}} + \frac{0.33^2}{2}\times\frac{8}{12}}{0.33\sqrt{8/12}} = 0.3908,\ d_2 = 0.3908 - 0.33\sqrt{8/12} = 0.1214,$$

$N(-d_1) = 0.3483$, $N(-d_2) = 0.4522$.

The price of the gap option is $28.5e^{-0.1\times 8/12} \times 0.4522 - 28.0656 \times 0.3483 = 2.2813$.

18. The expected payoff is $E[(S(1) - 11)I(S(1) > 10)] = E[S(1)\, I(S(1) > 10)] - 11P(S(1) > 10)$. From statements (ii) and (iii), we have $\alpha = 0.24$ and $\sigma = 0.2$. So,

$$\hat{d}_1 = \frac{\ln\frac{10}{10} + (0.24 + \frac{0.04}{2})}{0.2} = 1.3,\ \hat{d}_2 = 1.3 - 0.2 = 1.1.$$

$E[S(1)\, I(S(1) > 10)] - 11P(S(1) > 10)$

$= E[S(1)]\, N(\hat{d}_1) - 11\, N(\hat{d}_2)$

$= 10e^{0.24} \times 0.9032 - 11 \times 0.8643$

$= 1.9746$

19. The payoff of the gap option is

$[38 - S(7/12)]I(S(7/12) < 40)$

$= [40 - S(7/12) - 2]\ I(S(7/12) < 40)$

$= [40 - S(7/12)]\ I(S(7/12) < 40) - 2I(S(7/12) < 40)$

which is the difference between the payoff of an ATM put and 2 times the payoff of an ATM cash-or-nothing put.

By put-call parity,

$$3.45 - p(S(0), 40, 7/12) = 40 - e^{-0.025} - e^{-0.05} - 40e^{-0.1\times 7/12},$$

giving $p(S(0), 40, 7/12) = 3.12$.

The price of the gap put is $3.12 - 2(0.53) = 2.05$.

20. Method 1:

Since the payoffs of an ATM cash-or-nothing call and an ATM cash-or-nothing put sums to 1, and the present value of 1 is e^{-rT},

price of an ATM cash-or-nothing call is $e^{-rT} - 0.432 = e^{-0.1} - 0.432 = 0.472837$.

Then, the gap call price is

price of an asset-or-nothing call (45-strike) $- 46(0.472837) = 5.633$.

This gives

price of an asset-or-nothing call (45-strike) $= 27.3835$

Since the payoff of a 45-strike asset-or-nothing call and a 45-strike asset-or-nothing put sums to $S(T)$,

price of a 45-strike asset-or-nothing put is $45e^{-0.05} - 27.3835 = 15.4218$.

Finally, the price of the gap put is $43(0.432) - 15.4218 = 3.1542$.

Method 2:

The payoff of the gap option is

$[43 - S(1)]\mathrm{I}(S(1) \le 45)$

$= [46 - S(1) - 3]\ \mathrm{I}(S(1) \le 45)$

$= -[S(1) - 46 + 3]\ [1 - \mathrm{I}(S(1) > 45)]$

$= -[S(1) - 43] + [S(1) - 46]\mathrm{I}(S(1) > 45) + 3\mathrm{I}(S(1) > 45)$

$= -S(1) + 43 + [S(1) - 46]\mathrm{I}(S(1) > 45) + 3[1 - \mathrm{I}(S(1) \le 45)]$

$= -S(1) + 46 + [S(1) - 46]\mathrm{I}(S(1) > 45) - 3\mathrm{I}(S(1) \le 45)$

The price of the gap put is

$-S(0)e^{-\delta} + 46e^{-r} + 5.633 - 3(0.432) = -45e^{-0.05} + 46e^{-0.1} + 5.633 - 3(0.432) = 3.1542.$