Preface

Thank you for choosing ACTEX.

Since Exam MFE was introduced in May 2007, there have been quite a few changes to its syllabus and its learning objectives. To cope with these changes, ACTEX decided to launch a brand new study manual, which adopts a completely different pedagogical approach.

The most significant difference is that this edition is fully self-contained, by which we mean that, with this manual, you do not even have to read the "required" text (*Derivatives Markets* by Robert L. McDonald). By reading this manual, you should be able to understand the concepts and techniques you need for the exam. Sufficient practice problems are also provided in this manual. As such, there is no need to go through the textbook's end-of-chapter problems, which are either too trivial (simple substitutions) or too computationally intensive (Excel may be required). Note also that the textbook's end-of-chapter problems are not at all similar (in difficulty and in format) to the questions released by the Society of Actuaries (SoA).

We do not want to overwhelm students with verbose explanations. Whenever possible, concepts and techniques are demonstrated with examples and integrated into the practice problems.

Another distinguishing feature of this manual is that it covers the exam materials in a different order than it occurs in *Derivatives Markets*. There are a few reasons for using an alternative ordering:

1. Some topics are repeated quite a few times in the textbook, making students difficult to fully understand them. For example, "estimation of volatility" is discussed four times in *Derivatives Markets* (Sections 11.4, 12.5, 18.5, 23.2)! In sharp contrast, our study manual presents this topic fully in one single section (Module 3, Lesson 4.1).

2. The focus of the textbook is somewhat different from what SoA expects from the candidates. According to the SoA, the purpose of the exam is to develop candidates' knowledge of the theoretical basis. Nevertheless, the first half of the textbook is almost entirely devoted to applications. Therefore, we believe that reading the textbook or following the textbook's ordering is not the best use of your precious time.

Perhaps you have passed some SoA exams by memorizing formulas. However, from the released exam questions, you can easily tell that it is difficult, if not impossible, to pass Exam MFE/3F simply by memorizing all formulas in the textbook. In this connection, in this writing, we help you really learn the materials. By having the reasoning skills, you will discover that there is not really much to memorize.

To help you better prepare for the exam, we intentionally write the practice problems and the mock exams in a similar format as the released exam and sample questions. This will enable you to, for example, retrieve information more quickly in the real exam. Further, we have integrated the sample and previous exam questions provided by the SoA into the study manual into our examples, our practice problems, and our mock exams. This seems to be a better way to learn how to solve those questions, and of course, you will need no extra time to review those questions.

Our recommended procedure for use of this study manual is as follows:

1. Read the lessons in order.
2. Immediately after reading a lesson, complete the practice problems we provide for that lesson. Make sure that you understand every single practice problem.
3. After studying all 25 lessons, work on the mock exams.

If you find a possible error in this manual, please let us know at the “Customer Feedback” link on the ACTEX homepage (www.actexmadriver.com). Any confirmed errata will be posted on the ACTEX website under the “Errata & Updates” link.

A Note on Rounding and Using the Normal Table

To achieve the desired accuracy, we recommend that you store values in intermediate steps in your calculator. If you prefer not to, please keep at least six decimal places.

In this study guide, normal probability values and z-values are based on the standard normal distribution table, which is provided on page T0-3.

When using the standard normal distribution table, do not interpolate.

- Use the nearest z-value in the table to find the probability. Example: Suppose that you are to find $\Pr(Z < 0.759)$, where Z denotes a standard normal random variable. Because the z-value in the table nearest to 0.759 is 0.76, your answer is $\Pr(Z < 0.76) = 0.7764$.
- Use the nearest probability value in the table to find the z-value. Example: Suppose that you are to find z such that $\Pr(Z < z) = 0.7$. Because the probability value in the table nearest to 0.7 is 0.6985, your answer is 0.52.

Syllabus Reference

Module 0 and Module 1

Our Study Manual	The Required Text
Module 0: Review	
0.1	1.4, 5.2 (p.132 only)
0.2	5.1 – 5.3
0.3	2.2, 9.1
Module 1: Risk-Neutral Valuation in Discrete-time	
Lesson 1: Introduction to Binomial Trees	
1.1.1	10.1 (up to the middle of p.318)
1.1.2	10.1 (from the middle of p.318 to the middle of p.319)
1.1.3	10.1 (from p.320 to the middle of p.321)
Lesson 2: Multiperiod Binomial Trees	
1.2.1	10.2, 10.3
1.2.2	10.4
1.2.3	10.1 (from the middle of p.321), 11.3 (from the middle of p.358 to the middle of p.359)
Lesson 3: Options on Other Assets	
1.3.1	10.5 (up to p.331)
1.3.2	10.5 (p.332, up to the middle), 9.1 (formula 9.4 only)
1.3.3	10.5 (from the middle of p.332 to the middle of p.334)
Lesson 4: Pricing with True Probabilities	
1.4.1	11.2 (up to the middle of p.347)
1.4.2	11.2 (second half of p.347 to p.350)
Lesson 5: State Prices	
1.5.1	Appendix 11.B
1.5.2	Sample questions #27
1.5.3	Appendix 11.B

Module 2

Module 2: Risk-Neutral Valuation in Continuous-time	
Lesson 1: Brownian Motions	
2.1.1	11.3 (up to the beginning of p.353)
2.1.2	20.2 (up to the first two lines of p.651)
2.1.3	20.2 (from the bottom of p.653 to the bottom of p.654)
2.1.4	20.3 (up to the middle of p.656)
Lesson 2: Stochastic Calculus	
2.2.1	Scattered in 20.2 and 20.3
2.2.2	20.6 (excluding multivariate Ito's lemma)
2.2.3	Mainly scattered in 20.2 and 20.3, Example 20.1
2.2.4	20.2 (p.651 to p.653)
Lesson 3: Modeling Stock Price Dynamics	
2.3.1	18.1, 18.2
2.3.2	20.1, 18.3
2.3.3	18.4 (up to the end of p.602)
2.3.4	18.4 (from the bottom of p.602 to formula 18.30)
2.3.5	11.3 (from the middle of p.353 to p.354)
Lesson 4: The Sharpe Ratio and the Black-Scholes Equation	
2.4.1	12.1 (p.379)
2.4.2	20.4, 21.2 (p.682, but we generalize the approach here)
2.4.3	21.2 (from p.683 to the middle of p.688)
2.4.4	20.5, 21.3 (except "the backward equation"), 20.7 (except "finding the lease rate" and "valuing a claim on S^aQ^b")

Module 3

Module 3: The Black-Scholes Formula	
Lesson 1: Introduction to the Black-Scholes Formula	
3.1.1	22.1 (up to the middle of p.706)
3.1.2	22.1 (p.706 "ordinary options and gap options"), 12.1 (up to p.378)
3.1.3	12.2
Lesson 2: Greek Letters and Elasticity	
3.2.1	12.3 (p.382 – 385, p.387 up to middle, p.388 "Greek measures for portfolio"), 13.4 (up to p.426)
3.2.2	12.3 (p.386, middle of p.388)
3.2.3	12.3 (starting from p.389 to the end of the section 12.3)
3.2.4	Appendix 13.B
Lesson 3: Risk Management Techniques	
3.3.1	13.2, 13.3 (up to the next-to-last paragraph on p.419)
3.3.2	13.4 (p.427 to p.429)
3.3.3	13.3 (the bottom of p.419 to the end of the section), p.431
3.3.4	13.5 (the bottom of p.433 to the end of the section)
Lesson 4: Estimation of Volatilities and Expected Rates of Appreciation	
3.4.1	12.5, 23.1, 11.4, 23.2 (up to the middle of p.746), 18.5
3.4.2	18.6

Module 4

Module 4: Further Topics on Option Pricing	
Lesson 1: Exotic Options I	
4.1.1	14.2
4.1.2	Exercise 14.20
4.1.3	14.3
4.1.4	14.4 (except p.455 and example 14.2)
Lesson 2: Exotic Options II	
4.2.1	14.6
4.2.2	9.2
4.2.3	Exercise 14.21
4.2.4	14.5, 22.1 (p.706 “Ordinary options and gap options”)
Lesson 3: Simulation	
4.3.1	19.2, 19.3
4.3.2	19.4
4.3.3	19.5
Lesson 4: General Properties of Options	
4.4.1	9.3 (from p.299 to the first 2 lines on p.304)
4.4.2	9.3 (p.293 – 294 “European versus American options” and “maximum and minimum option prices”)
4.4.3	9.3 (p.297 “time to expiration”)
Lesson 5: Early Exercise for American Options	
4.5.1	9.3 (p.294 to the third paragraph of p.296), 11.1
4.5.2	9.3 (p.296 “Early exercise for puts”)
4.5.3	14.4 (p.455 and Example 14.2)

Module 5

Module 5: Interest Rate Models	
Lesson 1: Binomial Interest Rate Trees	
5.1.1	(Scattered in Chapter 7)
5.1.2	24.4
5.1.3	24.5 (up to Figure 24.9 on p.806)
Lesson 2: The Black Model	
5.2.1	9.1 (p.286 “Options on bonds”)
5.2.2	12.2 (p.381 “Options on futures”)
5.2.3	24.3
Lesson 3: An Equilibrium Equation for Interest Rate Derivatives	
5.3.1	24.1 (p.781 “An equilibrium equations for bonds up to (24.17)”)
5.3.2	24.1 (p.783 the first 2 paragraphs)
5.3.3	24.1 (p.783 to the middle of p.784)
5.3.4	24.1 (the middle of p.783 to the end of section 24.1)
Lesson 4: The Rendleman-Bartter, Vasicek and Cox-Ingersoll-Ross Model	
5.4.1	24.2 (p.785 “The Rendleman-Bartter model)
5.4.2	24.2 (p.786 “The Vasicek model”)
5.4.3	24.2 (p.787 “The Cox-Ingersoll-Ross model” and “Comparing Vasicek and CIR”)

Sample pages from Module 2 Lesson 2

This lesson contains 24 pages. We are showing pages 1 to 6.

Lesson 2 Stochastic Calculus

OBJECTIVES

1. To understand stochastic differential equations and diffusion processes
2. To use Itô's lemma to obtain the stochastic differential equation for a function of a diffusion process
3. To solve three types of stochastic differential equations
4. To understand the concept of variations of Brownian motions

2. 2. 1 Stochastic Differential Equations

Differentials

Suppose that the rate of change in $x(t)$ depends on the time t and the value of $x(t)$ itself. That is,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \mathrm{f}\,(t, x(t)).$$

We can interpret this equation by using the concept of differentials. We "multiply" both sides of the equation by dt to obtain

$$\mathrm{d}x(t) = \mathrm{f}(t, x(t))\mathrm{d}t,$$

which says that the change in x over a very short time interval $[t, t + \mathrm{d}t]$ is given by $\mathrm{f}(t, x(t))\mathrm{d}t$.

To illustrate, we apply the above to a bank account crediting a constant force of interest r. Suppose that you put \$1 into the bank account at time 0 and that $a(t)$ is the accumulated value at time t. Consider a very short time interval $[t, t + \mathrm{d}t]$.

(1) Noting that r is the interest rate per annum, the interest rate credited in $[t, t + \mathrm{d}t]$ is $r\mathrm{d}t$.

(2) The dollar amount of interest earned in $[t, t + \mathrm{d}t]$ is $a(t) \times (r\mathrm{d}t)$.

(3) Thus, the change in the bank account is $ra(t)\mathrm{d}t$.

In other words,

$$\mathrm{d}a(t) = ra(t)\mathrm{d}t \quad \text{or} \quad \frac{\mathrm{d}a(t)}{\mathrm{d}t} = ra(t).$$

Stochastic differential equations

In financial markets, not all variables are deterministic. What if the change in x is perturbed by a standard Brownian motion? Suppose that

$$\mathrm{d}X(t) = a(t, X(t))\mathrm{d}t + b(t, X(t))\mathrm{d}Z(t),$$

or in shorthand notation

$$\mathrm{d}X = a\mathrm{d}t + b\mathrm{d}Z.$$

The above is called a **stochastic differential equation** (SDE) and X is said to be a diffusion. In this equation, $\mathrm{d}Z(t)$ is the change in the standard Brownian motion over $[t, t + \mathrm{d}t]$, while $\mathrm{d}X(t)$ is the change in X over $[t, t + \mathrm{d}t]$. Intuitively, you may view them as $Z(t + \mathrm{d}t) - Z(t)$ and $X(t + \mathrm{d}t) - X(t)$.

To interpret a SDE, you need to know the following.

(1) Since $\mathrm{d}Z(t)$ is random, $\mathrm{d}X(t)$, and hence $X(t)$, are random. That is why we have used an uppercase letter for X.

(2) The distribution of $\mathrm{d}Z(t) = Z(t + \mathrm{d}t) - Z(t)$ is $\mathrm{N}(0, \mathrm{d}t)$. Hence, $\mathrm{E}[\mathrm{d}Z(t)] = 0$ and $\mathrm{Var}[\mathrm{d}Z(t)] = \mathrm{d}t$.

(3) It follows from independent increments that $\mathrm{d}Z(t)$ is independent of the history $\{Z(u): 0 \le u \le t\}$. In particular, $\mathrm{d}Z(t)$ and $Z(t)$ are independent.

(4) Given the value of $X(t)$, the terms $a(t, X(t))$ and $b(t, X(t))$ are no longer random, and hence

$$\mathrm{E}[\mathrm{d}X(t) \mid X(t)] = a(t, X(t))\mathrm{d}t \quad \text{and} \quad \mathrm{Var}[\mathrm{d}X(t) \mid X(t)] = b^2(t, X(t))\mathrm{d}t$$

By virtue of (4), $a(x, t)$ and $b(x, t)$ are called the **drift** and **volatility** of the SDE. If $a(x, t) = 0$, then $X(t)$ is said to be driftless.

2. 2. 2 Itô's Lemma

Suppose that $Y(t)$ is a function of time t and a standard Brownian motion $Z(t)$. Then the change in Y, $dY(t)$, should be related to t, $Z(t)$ and $dZ(t)$. We are interested in obtaining the SDE for Y. This can be accomplished by using Itô's lemma.

FORMULA

Itô's Lemma (simplified version)

Let $Y(t) = f(t, Z(t))$. Then

$$dY(t) = f_t(t, Z(t))dt + f_z(t, Z(t))dZ(t) + \frac{1}{2}f_{zz}(t, Z(t))[dZ(t)]^2,$$

where $[dZ(t)]^2 = dt$.

Thus, the drift is $f_t(t, Z(t)) + \frac{1}{2}f_{zz}(t, Z(t))$ and the volatility is $f_z(t, Z(t))$.

To derive the SDE for $Y(t) = f(t, Z(t))$, use the following procedure.

Step 1: Recognize the function $f(t, z)$. This can be done by replacing all $Z(t)$ by z.

Step 2: Find the three partial derivatives f_t, f_z and f_{zz}.

Step 3: Plug f_t, f_z and f_{zz} into Itô's lemma. Remember that there is a $\frac{1}{2}$ attached to f_{zz}.

Step 4: Use $[dZ(t)]^2 = dt$ and collect like terms to get the drift and volatility.

Read the following example.

Example 2.2.1

Let $Z(t)$ be a standard Brownian motion. Find $dY(t)$ for the following:

(a) $Y(t) = Z^2(t)$

(b) $Y(t) = tZ^2(t)$

Solution

(a) *Step 1:* Since $Y(t) = f(t, Z(t)) = Z^2(t)$, by replacing $Z(t)$ by z, we have $f(t, z) = z^2$.

Step 2: $f_t(t, z) = 0$, $f_z(t, z) = 2z$, $f_{zz}(t, z) = 2$.

Step 3: By Itô's lemma,

$$dY(t) = 0\,dt + 2Z(t)dZ(t) + \frac{1}{2}(2)[dZ(t)]^2 .$$

Step 4: By using $[dZ(t)]^2 = dt$, we get

$$dY(t) = 2Z(t)dZ(t) + \frac{1}{2}(2)dt = dt + 2Z(t)dZ(t) .$$

(b) *Step 1:* Since $Y(t) = f(t, Z(t)) = tZ^2(t)$, by replacing $Z(t)$ by z, we have $f(t, z) = tz^2$.

Step 2: $f_t(t, z) = z^2$, $f_z(t, z) = 2tz$, $f_{zz}(t, z) = 2t$.

Step 3: By Itô's lemma,

$$dY(t) = Z^2(t)dt + 2tZ(t)dZ(t) + \frac{1}{2}(2t)[dZ(t)]^2 .$$

Step 4: By using $[dZ(t)]^2 = dt$, we get

$$dY(t) = Z^2(t)dt + 2tZ(t)dZ(t) + \frac{1}{2}(2t)dt = [t + Z^2(t)]dt + 2tZ(t)dZ(t) .$$

[END]

Now let us find the SDEs satisfied by ABMs and GBMs introduced in Lesson 1 of this module.

Example 2.2.2

Let $Z(t)$ be a standard Brownian motion and $Y(0)$ be a constant. Find $dY(t)$ for the following:

(a) $Y(t) = Y(0) + \mu t + \sigma Z(t)$

(b) $Y(t) = Y(0)\exp\left[(\mu - \frac{\sigma^2}{2})t + \sigma Z(t)\right]$

(Note: Here we use $Y(t) = Y(0)\exp\left[(\mu - \frac{\sigma^2}{2})t + \sigma Z(t)\right]$ but not $Y(t) = Y(0)\exp[\mu t + \sigma Z(t)]$.

The reason will become clear in the next section.)

Solution

(a) *Step 1:* Since $Y(t) = f(t, Z(t)) = Y(0) + \mu t + \sigma Z(t)$, by replacing $Z(t)$ by z, we have $f(t, z) = Y(0) + \mu t + \sigma z$.

Step 2: $f_t(t, z) = \mu$, $f_z(t, z) = \sigma$, $f_{zz}(t, z) = 0$.

Step 3: By Itô's lemma,

$$dY(t) = \mu\, dt + \sigma\, dZ(t) + \frac{1}{2}(0)[dZ(t)]^2.$$

Step 4: A further simplification is not needed. The final answer is

$$dY(t) = \mu\, dt + \sigma\, dZ(t).$$

(b) *Step 1:* Since $Y(t) = f(t, Z(t)) = Y(0)\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma Z(t)}$, by replacing $Z(t)$ by z,

we have $f(t, z) = Y(0)\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma z}$.

Step 2: $f_t(t, z) = Y(0)(\mu - \frac{\sigma^2}{2})e^{(\mu - \frac{\sigma^2}{2})t + \sigma z}$, $f_z(t, z) = Y(0)\sigma\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma z}$,

$f_{zz}(t, z) = Y(0)\sigma^2\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma z}$.

Step 3: By Itô's lemma,

$$dY(t) = Y(0)(\mu - \frac{\sigma^2}{2})e^{(\mu - \frac{\sigma^2}{2})t + \sigma Z(t)}\, dt + Y(0)\sigma\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma Z(t)}\, dZ(t)$$
$$+ \frac{1}{2}Y(0)\sigma^2 e^{(\mu - \frac{\sigma^2}{2})t + \sigma Z(t)}[dZ(t)]^2.$$

Step 4: By using $[dZ(t)]^2 = dt$, we see that the last term can cancel out the latter part resulting from the expansion of the first term. Thus,

$$dY(t) = Y(0)\, \mu e^{(\mu - \frac{\sigma^2}{2})t + \sigma Z(t)}\, dt + Y(0)\sigma\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma Z(t)}\, dZ(t).$$

Finally, by using $Y(0)\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma Z(t)} = Y(t)$, we get

$$dY(t) = \mu Y(t) dt + \sigma Y(t) dZ(t).$$

[END]

Now suppose that $Y(t)$ is a function of time t and another function $X(t)$. Suppose also that we are given the SDE for $X(t)$:

$$dX(t) = a(t, X(t))dt + b(t, X(t))dZ(t).$$

To find the SDE for Y, we use the general version of Itô's lemma.

FORMULA

Itô's Lemma (general version)

Let $Y(t) = \mathrm{f}(t, X(t))$. Then

$$\mathrm{d}Y(t) = \mathrm{f}_t(t, X(t))\mathrm{d}t + \mathrm{f}_x(t, X(t))\mathrm{d}X(t) + \frac{1}{2}\mathrm{f}_{xx}(t, X(t))[\mathrm{d}X(t)]^2,$$

where $[\mathrm{d}X(t)]^2 = b^2(t, X(t))\mathrm{d}t$.

Note that the simplified version of Itô's lemma is the special case when $a(t, x) = 0$ and $b(t, x) = 1$.

To use of Itô's lemma for a function of t and $X(t)$, follow the procedure below.

Step 1: Recognize the function $\mathrm{f}(t, x)$. This can be done by replacing all $X(t)$ by x.

Step 2: Find the three partial derivatives f_t, f_x and f_{xx}.

Step 3: Plug f_t, f_x and f_{xx} into Itô's lemma. Remember that there is a $\frac{1}{2}$ attached to f_{xx}.

Step 4: Use the SDE of X, $[\mathrm{d}X(t)]^2 = b^2(t, X(t))\mathrm{d}t$ and collect like terms to get the drift and volatility.

Example 2.2.3

Suppose that $\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}Z(t)$. Find $\mathrm{d}Y(t)$ for $Y(t) = \ln X(t)$.

Solution

Step 1: Since $Y(t) = \mathrm{f}(t, X(t)) = \ln X(t)$, by replacing $X(t)$ by x, we have $\mathrm{f}(t, x) = \ln x$.

Step 2: $\mathrm{f}_t(t, x) = 0$, $\mathrm{f}_x(t, x) = \frac{1}{x}$, $\mathrm{f}_{xx}(t, x) = -\frac{1}{x^2}$.

Step 3: By Itô's lemma,

$$\mathrm{d}Y(t) = 0\,\mathrm{d}t + \frac{1}{X(t)}\mathrm{d}X(t) + \frac{1}{2}\left(-\frac{1}{X^2(t)}\right)[\mathrm{d}X(t)]^2.$$

Step 4: We have $a(t, X(t)) = \mu X(t)$ and $b(t, X(t)) = \sigma X(t)$, so that $[\mathrm{d}X(t)]^2 = \sigma^2 X^2(t)\mathrm{d}t$.

$$\mathrm{d}Y(t) = \frac{1}{X(t)}[\mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}Z(t)] + \frac{1}{2}\left(-\frac{1}{X^2(t)}\right)\sigma^2 X^2(t)\mathrm{d}t = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \mathrm{d}Z(t).$$

[END]

Sample pages from Module 2 Lesson 4

This lesson contains 33 pages. We are showing pages 1 to 8.

Lesson 4 The Sharpe Ratio and the Black-Scholes Equation

OBJECTIVES

1. To state the meaning of the Black-Scholes framework
2. To understand the concept of the Sharpe ratio and the restriction it imposes on the drift and volatility of a contingent claim
3. To understand the Black-Scholes equation and its uses
4. To understand the concept of risk-neutral valuation
5. To recognize the relation between the true and the risk-neutral measures

2. 4. 1 The Black-Scholes Framework

In this lesson we study the concept of risk-neutral valuation in continuous-time. This lesson is the most theoretical in the exam syllabus. Because questions from this lesson tends to be tricky, it is important to fully grasp the concepts taught in this lesson.

In exam MFE, you would see statements such as "Assume the Black-Scholes framework" and "Suppose that S follows the Black-Scholes model" very often.

By the Black-Scholes framework, we mean the following:

- The underlying asset $S(t)$ follows a geometric Brownian motion.
- The underlying asset is either nondividend-paying or pays dividends continuously at a level proportional to its price.
- The risk-free interest rate is constant.
- There are no transaction cost or taxes.

– It is possible to purchase or short-sell any units of the underlying asset.

– The borrowing rate and the lending rate are both equal to the risk-free interest rate.

– There are no arbitrage opportunities.

The most important assumption is of course the first one: GBM. A lot of things can be said on this assumption and you should now understand why we have spent three lessons (in particular Lesson 3) on GBMs. To recap, recall that the five statements below are equivalent.

(1) $S(t)$ is a GBM with drift $\alpha - \delta$ and volatility σ.

(2) $\dfrac{\mathrm{d}S(t)}{S(t)} = (\alpha - \delta)\mathrm{d}t + \sigma\,\mathrm{d}Z(t)$ where $Z(t)$ is a standard Brownian motion.

(3) $\mathrm{d}[\ln S(t)] = (\alpha - \delta - \dfrac{\sigma^2}{2})\mathrm{d}t + \sigma\,\mathrm{d}Z(t)$.

(4) $S(t) = S(0)\,e^{(\alpha-\delta-\frac{\sigma^2}{2})t+\sigma Z(t)}$.

(5) $\ln S(t)$ is normal distributed with mean $\ln S(0) + (\alpha - \delta - \dfrac{\sigma^2}{2})t$ and variance $\sigma^2 t$.

2. 4. 2 The Sharpe Ratio

Let $X(t)$ be the price of an asset. The Sharpe ratio of X at time t is defined as the ratio of the instantaneous average risk premium to the instantaneous volatility. Heuristically, the Sharpe ratio is a measure of the risk-return trade off.

We assume the following:

(1) The dynamics of $X(t)$ follow

$$\frac{\mathrm{d}X(t)}{X(t)} = m\mathrm{d}t + s\mathrm{d}Z(t).$$

(Warning: We are not using $\mathrm{d}X(t) = m\mathrm{d}t + s\mathrm{d}Z(t)$!)

Here $m = m(X(t), t)$ and $s = s(X(t), t)$ can depend on t and the time-t price of X. For simplicity, we suppress the arguments $X(t)$ and t.

(2) The asset pays dividends continuously at a rate proportional to its price. The continuous dividend yield is δ. (We have $\delta = 0$ if the asset is nondividend-paying.) The dollar amount of dividend over an infinitesimally short time period $(t, t + \mathrm{d}t)$ is

$$X(t)\,\delta\mathrm{d}t.$$

Then,

- $\frac{dX(t)}{X(t)}$ is the instantaneous return due to capital gains,
- $m\,dt = E\left[\frac{dX(t)}{X(t)}\middle| X(t)\right]$ is the expected instantaneous return due to capital gains,
- $s^2\,dt = \text{Var}\left[\frac{dX(t)}{X(t)}\middle| X(t)\right]$ is the variance of the instantaneous return due to capital gains.

The total return on X is the sum of capital gains and dividends. As a result, the instantaneous total return is $m + \delta$, and the instantaneous risk premium is $m + \delta - r$. The Sharpe ratio is defined by the ratio of the instantaneous risk premium to the instantaneous standard deviation.

F O R M U L A

The Sharpe Ratio of an Asset

If $\frac{dX(t)}{X(t)} = m dt + s dZ(t)$ and the continuous dividend yield is δ, then the Sharpe ratio is

$$\phi = \frac{m+\delta-r}{s}.$$

Read the following examples.

Example 2.4.1

Consider the Black-Scholes model for a stock price $S(t)$. Find the Sharpe ratio of S.

Solution

(1) The dynamics of S is

$$\frac{dS(t)}{S(t)} = (\alpha - \delta)\,dt + \sigma\,dZ(t).$$

(2) S pays dividends continuously at a constant rate proportional to its price. The dividend yield is δ.

Thus the Sharpe ratio of S is $\phi = \frac{(\alpha-\delta)+\delta-r}{\sigma} = \frac{\alpha-r}{\sigma}$, which is a constant because α and σ are both constants.

[END]

Example 2.4.2

Consider again the Black-Scholes model for a stock price $S(t)$. Let $V(S(t), t) = F_{t,T}^{P}(S)$ be the time-t prepaid forward price for the delivery of one share of S at time T. Find the Sharpe ratio of this contract.

Solution

(1) We need to derive the dynamics of V.

Since $V(S(t), t) = F_{t,T}^{P}(S) = e^{-\delta(T-t)} S(t)$, we use Itô's lemma.

Let $f(s, t) = se^{-\delta(T-t)}$. Then $f_t(s, t) = \delta se^{-\delta(T-t)}$, $f_s(s, t) = e^{-\delta(T-t)}$, $f_{ss}(s, t) = 0$.

$$\begin{aligned} dV(S(t), t) &= \delta\, S(t)\, e^{-\delta(T-t)}dt + e^{-\delta(T-t)}\, dS(t) \\ &= \delta e^{-\delta(T-t)}\, S(t)\, dt + e^{-\delta(T-t)}\, [(\alpha - \delta)S(t)dt + \sigma S(t)dZ(t)] \\ &= e^{-\delta(T-t)}\, [\alpha\, S(t)dt + \sigma S(t)dZ(t)] \\ &= V(S(t), t)\, [\alpha\, dt + \sigma\, dZ(t)] \end{aligned}$$

So, the dynamics is $\dfrac{dV(S(t),t)}{V(S(t),t)} = \alpha\, dt + \sigma\, dZ(t)$. In this case, $m = \alpha$, and $s = \sigma$.

(2) A prepaid forward contract pays no dividend. The dividend yield is 0.

Thus, the Sharpe ratio is $\phi = \dfrac{m + \delta - r}{s} = \dfrac{\alpha - r}{\sigma}$.

[END]

We observe from Example 2.4.1 and Example 2.4.2 that both the stock and the prepaid forward contract have the same Sharpe ratio. Is it a coincidence? Actually it is not. The reason for the equality of Sharpe ratio is that both the stock and the prepaid forward contract have the same underlying source of risk: it is the same $Z(t)$ that causes the stock price and prepaid forward price to change over time randomly.

FORMULA

Equality of Sharpe Ratios

The Sharpe ratios of two assets driven by the same Brownian motion must be the same.

In particular, any contingent claim written on a stock that follows a GBM must have a Sharpe ratio of $\phi = \dfrac{\alpha - r}{\sigma}$.

We need to study the proof of the above theorem, because the proof can be (and has been) tested in various ways. The idea of the proof involves **hedging**, an important concept that we will explore in more detail in Module 3. This is the only proof in this lesson. So please read it.

Proof:

(1) Set-up:

Two risky assets X and Y, with price processes

$$\frac{dX(t)}{X(t)} = m_X dt + s_X dZ(t), \qquad \frac{dY(t)}{Y(t)} = m_Y dt + s_Y dZ(t).$$

The continuous dividend yield rates for X and Y are δ_X are δ_Y, respectively.

Notice that here s_X and s_Y can be negative. This does not mean that the volatility of the return is negative, but that the return moves in a direction that is opposite to $Z(t)$.

(2) Hedging:

Suppose that we have 1 unit of X at time t. Our goal is to purchase / sell appropriate units of Y and cash, so that we have an instantaneously **risk-free** and **costless** portfolio.

Suppose we purchase N units of Y and invest W dollars at risk-free interest rate r.

Then the value of the portfolio at time t is $V(t) = X(t) + NY(t) + W$.

To make the portfolio costless, W should be chosen so that $W = -X(t) - NY(t)$.

What would happen after an infinitesimally short period dt?

– The change in the price of X and Y are $dX(t)$ and $dY(t)$.

– The amount of dividends generated from 1 unit of X and Y are $X(t)\delta_X dt$ and $Y(t)\delta_Y dt$.

– The interest earning for 1 dollar is rdt.

So,

$$\begin{aligned} dV(t) &= dX(t) + X(t)\delta_X dt + N(dY(t) + Y(t)\delta_Y dt) + rWdt \\ &= (m_X X(t) + \delta_X X(t) + Nm_Y Y(t) + N\delta_Y Y(t) + rW)dt + [s_X X(t) + Ns_Y Y(t)]dZ(t) \end{aligned}$$

To make the portfolio instantaneously risk-free, we pick N so that $s_X X(t) + Ns_Y Y(t) = 0$, or equivalently,

$$N = -\frac{s_X X(t)}{s_Y Y(t)}.$$

As a result, $W = -X(t) - \left(-\frac{s_X X(t)}{s_Y Y(t)}\right)Y(t) = -\left(1 - \frac{s_X}{s_Y}\right)X(t)$.

(3) Equating the drift of the hedged portfolio to 0:

By picking N and W as in (2), V is instantaneously risk-free and costless. What should such a portfolio earn? The return on a risk-free investment of zero dollars must be zero, or otherwise there will be an arbitrage! So, the drift of dV must be 0:

$$m_X X(t) + \delta_X X(t) + N m_Y Y(t) + N\delta_Y Y(t) + rW = 0.$$

After some algebraic simplifications, we get

$$\frac{m_X + \delta_X - r}{s_X} = \frac{m_Y + \delta_Y - r}{s_Y}.$$

[END]

The formula for N can be written as

$$N = -\frac{s_X X(t)}{s_Y Y(t)} = -\frac{\text{volatility coefficient of } dX(t)}{\text{volatility coefficient of } dY(t)}. \quad (2.4.1)$$

This formula makes a lot of sense. Suppose that s_X and s_Y are positive. If $Z(t)$ increases, then the risky part of X and Y would both be positive. In order that they cancel each other, one should sell Y. If X is riskier (i.e., s_X is large relative to s_Y), then we need more units of Y for a complete cancellation of risk.

Example **2.4.3**

Consider two assets X and Y. There is a single source of uncertainty which is captured by a standard Brownian motion $\{Z(t)\}$. The prices of the assets satisfy the stochastic differential equations

$$\frac{dX(t)}{X(t)} = 0.07dt + 0.12dZ(t) \text{ and } d[\ln Y(t)] = Adt + 0.16dZ(t),$$

where A is a constant. You are also given that

(i) X is nondividend-paying;

(ii) Y pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(iii) The continuously compounded risk-free interest rate is 0.04.

Determine A.

Solution

(1) We compute the Sharpe ratio of Y. We first derive the dynamics of Y. By the equivalence of different representations of a GBM, we have

$$\frac{dY}{Y} = \left(A + \frac{0.16^2}{2}\right)dt + 0.16\,dZ(t).$$

So, the Sharpe ratio of Y is $\phi_Y = \dfrac{A + \dfrac{0.16^2}{2} + 0.03 - 0.04}{0.16} = \dfrac{A + 0.0028}{0.16}$.

(2) The Sharpe ratio of X is $\phi_X = \dfrac{0.07+0-0.04}{0.12} = 0.25$.

By equating the Sharpe ratios, $\dfrac{A+0.0028}{0.16} = 0.25$, we have

$$A = -0.0028 + 0.25(0.16) = 0.0372.$$

[END]

Example **2.4.4**

The prices of two stocks are governed by:

$$\frac{\mathrm{d}X(t)}{X(t)} = 0.06\mathrm{d}t + 0.02\mathrm{d}Z(t), \quad \frac{\mathrm{d}Y(t)}{Y(t)} = 0.09\mathrm{d}t + k\mathrm{d}Z(t).$$

where $Z(t)$ is a standard Brownian motion and k is a constant. You are given:

(i) The current stock prices are $X(0) = 25$ and $Y(0) = 50$.

(ii) Both stocks pays dividends at a rate proportional to its price. The dividend yields of X and Y are $\delta + 0.01$ and δ, respectively.

(iii)The continuously compounded risk-free interest rate is 4%.

To construct a zero-investment, risk-free portfolio in which there are exactly 16 shares of X, one needs to trade a certain number of Y and borrow 100 dollars at risk-free rate. Find δ.

Solution

Equating the Sharpe ratios,

$$\frac{0.06+\delta+0.01-0.04}{0.02} = \frac{0.09+\delta-0.04}{k}.$$

Suppose one has 1 share of X, then the hedge portfolio has

$$N = -\frac{0.02(25)}{k(50)} = -\frac{1}{100k}$$

By (2.4.1)

units of Y and

$$W = -X(0) - NY(0) = -25 - 50\left(-\frac{1}{100k}\right) = -25 + \frac{1}{2k}$$

dollars of cash.

From the question, we know that in a portfolio with 16 shares of X, we need to borrow 100. So for a portfolio with one share of X, we have $W = -\dfrac{100}{16} = -6.25$. On solving $-25 + \dfrac{1}{2k} = -6.25$, we get $k = \dfrac{1}{37.5}$. Substituting the value of k into the equation for δ, we have

$$\frac{0.03+\delta}{0.02}=37.5(0.05+\delta),$$

which gives $\delta = 0.03$.

[END]

2. 4. 3 The Black-Scholes Equation

In Example 2.4.2, we calculated the Sharpe ratio for a prepaid forward contract. What if we apply the same procedure to a derivative on S?

We consider a derivative whose time-t price when the stock price is $S(t)$ is $V(S(t), t)$.

Step 1: By using Itô's lemma, we obtain $dV(S(t), t)$.

Step 2: Then by finding m and s, we obtain the Sharpe ratio of the derivative.

(It is not hard to see that

$$m=\frac{1}{V}\left[V_t+(\alpha-\delta)SV_s+\frac{1}{2}\sigma^2S^2V_{ss}\right],\quad s=\frac{SV_s}{V}\sigma.$$

But these two formulas are not important and you do not need to remember them now. We will revisit the formula for s in Module 3.)

Step 3: By setting the Sharpe ratio to $\dfrac{\alpha-r}{\sigma}$, we get the Black-Scholes equation.

F O R M U L A

The Black-Scholes Equation

$$\frac{\partial V}{\partial t}+(r-\delta)S\frac{\partial V}{\partial s}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial s^2}=rV$$

The pricing formula for any derivative must satisfy the Black-Scholes (BS) equation. So, the BS equation can be used as a polygraph: if you are presented a formula $V(S(t), t)$ that looks like the time-t price of something, you can tell if it is indeed the price of a certain derivative by checking if it satisfies the BS equation.

To familiar yourself with this funny concept, let us take a look at a toy example.

Sample pages from Module 3 Lesson 2

This lesson contains 33 pages. We are showing pages 1 to 10.

Lesson 2 Greek Letters and Elasticity

OBJECTIVES

1. To study Greek letters
2. To calculate the mean return and volatility of a derivative
3. To calculate the elasticity of a derivative

In this lesson, we focus on different measures of risk. These measures, as we will demonstrate in the next lesson, can help us hedge the risk associated with a portfolio of risky assets.

3. 2. 1 Greek Letters: Delta, Gamma and Theta

In the Black-Scholes framework, the price of any derivative security depends on the following six factors:

Stock	Option	Environment
Current Stock Price	Time	Risk-free Rate
Volatility	Payoff Feature	
Dividend Yield		

For example, for a power contract, we have

$$V(S, t) = S^a \exp[(-r + a(r - \delta) + 0.5a(a - 1)\sigma^2)(T - t)],$$

which is a function of the current stock price S, volatility σ, dividend yield δ, time t, payoff feature a, and the risk-free rate r. For a cash-or-nothing European call,

$$V(S,t) = e^{-r(T-t)}N(d_2), \quad d_2 = \frac{\ln\frac{S}{K} + (r - \delta - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

which is a function of the current stock price S, volatility σ, dividend yield δ, time t, payoff feature K, and the risk-free rate r.

As time proceeds,

- t increases,
- S changes.

Both would lead to a change in the price of a derivative.

One way to quantify the risk of a derivative is to measure how sensitive $V(S, t)$ is when S or t changes. The sensitivities can be estimated by the partial derivatives of V with respect to S and t.

FORMULA

Delta, Gamma and Theta

$$\Delta = \frac{\partial V}{\partial S}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S}, \quad \theta = \frac{\partial V}{\partial t}$$

The textbook gives the following verbal and non-rigorous interpretations of the first three Greek letters:

Delta (Δ) measures the change in the price of a derivative when the stock price increases by $1. A large Δ means that the derivative price is very sensitive to small changes in S. Therefore, a derivative is riskier if it has a larger Δ.

Gamma (Γ) measures the change in delta when the stock price increases by $1.

Theta (θ) measures the change in the price of a derivative when there is a **decrease** in the time to expiration ($T - t$), that is, an increase in t (as T is fixed).

Example 3.2.1

Assume the Black-Scholes framework. Compute the time-t delta and gamma for a cash-or-nothing call.

Solution

We have $V(S, t) = e^{-r(T-t)}N(d_2)$, where $d_2 = \dfrac{\ln\frac{S}{K} + (r - \delta - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$.

The time-t delta is

$$\Delta = \frac{\partial V(S,t)}{\partial S} = e^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S} = e^{-r(T-t)}N'(d_2)\frac{1}{S\sigma\sqrt{T-t}},$$

where $N'(z) = \dfrac{1}{\sqrt{2\pi}}e^{-z^2/2}$ is the pdf of N(0, 1).

The time-t gamma is

$$\begin{aligned}\Gamma &= \frac{\partial \Delta(S,t)}{\partial S} \\ &= e^{-r(T-t)}\frac{1}{S\sigma\sqrt{T-t}}N''(d_2)\frac{\partial d_2}{\partial S} + e^{-r(T-t)}N'(d_2)\frac{\partial}{\partial S}\left(\frac{1}{S\sigma\sqrt{T-t}}\right) \\ &= e^{-r(T-t)}\frac{N''(d_2)}{S^2\sigma^2(T-t)} - e^{-r(T-t)}N'(d_2)\frac{1}{S^2\sigma\sqrt{T-t}},\end{aligned}$$

where $N''(z) = \dfrac{1}{\sqrt{2\pi}}\dfrac{\mathrm{d}}{\mathrm{dz}}e^{-z^2/2} = -\dfrac{z}{\sqrt{2\pi}}e^{-z^2/2} = -zN'(z)$.

[END]

It is very tedious to compute the time-t theta of the cash-or-nothing call by differentiating $V(S, t)$ with respect to t. Rather than working directly on the partial derivative, we can calculate theta by using the Black-Scholes equation, which relates Δ, Γ, and θ as follows.

FORMULA

Relation between Delta, Gamma and Theta

$$\theta + (r - \delta)S\Delta + \frac{1}{2}\sigma^2 S^2\Gamma = rV$$

The following table shows the formulas for Δ, Γ, and θ of European calls and puts.

Greek	Call	Put
$\Delta = \dfrac{\partial V}{\partial S}$	$e^{-\delta(T-t)}N(d_1)$	$-e^{-\delta(T-t)}N(-d_1)$
$\Gamma = \dfrac{\partial^2 V}{\partial S^2}$	$\dfrac{e^{-\delta(T-t)}N'(d_1)}{S\sigma\sqrt{T-t}}$	same as Γ of call
$\theta = \dfrac{\partial V}{\partial t}$	$\delta S e^{-\delta(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2) - \dfrac{S\sigma e^{-\delta(T-t)}N'(d_1)}{2\sqrt{T-t}}$	call θ $+rKe^{-r(T-t)} - \delta S e^{-\delta(T-t)}$

You must remember the formulas for Δ for Exam MFE. However, the formulas for Γ and θ are optional. If you are asked to compute Γ, you can simply differentiate the formula for Δ. In the unlikely event that you are asked to compute θ, you should first compute V, Δ and Γ, and then use the Black-Scholes equation to solve for θ.

Example 3.2.2

Assume the Black-Scholes framework. You are given that:

(i) A stock S has a current price of 15.

(ii) The stock pays dividends continuously at a rate that is proportional to its price. The dividend yield is 4%.

(iii) The volatility of the stock is less than 0.3.

(iv) A 3-month at-the-money European put option on S has a delta of –0.4360.

(v) The continuously compounded risk-free interest rate is 8%.

Compute the price of the put option.

Solution

Let the current time point be $t = 0$. The delta of the put is

$$-e^{-\delta T}N(-d_1) = -0.4360$$

$$N(-d_1) = -0.4360e^{0.04/4} = -0.44038.$$

$$d_1 = 0.15$$

$$\frac{\ln\frac{15}{15} + (0.08 - 0.04 + \frac{\sigma^2}{2}) \times 0.25}{\sigma\sqrt{0.25}} = 0.15$$

$$0.125\sigma^2 - 0.075\sigma + 0.01 = 0$$

$$\sigma = 0.2 \text{ or } 0.4 \text{ (rejected)}$$

As a result, $d_2 = d_1 - \sigma\sqrt{T} = 0.15 - 0.2\sqrt{0.25} = 0.05$, and $N(-d_2) = 0.4801$.

The price of the put is $15e^{-0.08/4} \times 0.4801 - 15e^{-0.04/4} \times 0.44038 = 0.51893$.

[END]

There are a few special relations between the Greeks for calls and puts. These relations are derived from put-call parity:

$$c(S, t) - p(S, t) = Se^{-\delta(T-t)} - Ke^{-r(T-t)}.$$

Differentiating both sides of the put-call parity equation with respect to S, we get

$$\text{call delta} - \text{put delta} = e^{-\delta(T-t)}$$

Differentiating both sides of the put-call parity equation with respect to S twice, we get

$$\text{call gamma} - \text{put gamma} = 0$$

Differentiating both sides of the put-call parity equation with respect to t, we get

$$\text{call theta} - \text{put theta} = \delta Se^{-\delta(T-t)} - rKe^{-r(T-t)}.$$

Example **3.2.3**

Assume the Black-Scholes framework. You are given that:

(i) A nondividend-paying stock has a current price of 10 and a volatility of 40%.

(ii) A T-year K-strike European put option on S has a price of 2.4954 and a theta of –0.3903.

(ii) A T-year K-strike European call option written on S has a delta of 0.4480 and a gamma of 0.09889.

Find r, the continuously compounded risk-free interest rate.

Solution

Let the current time point be $t = 0$. Since the stock is nondivdend-paying, $\delta = 0$, and

$$\text{call delta} - \text{put delta} = e^{-\delta T} = 1$$

$$\Rightarrow \quad \text{put delta} = 0.4480 - 1 = -0.5520.$$

Moreover, since call gamma = put gamma, put gamma is 0.09889.

We now have the price, delta, gamma and theta of the put. The Black-Scholes equation says that

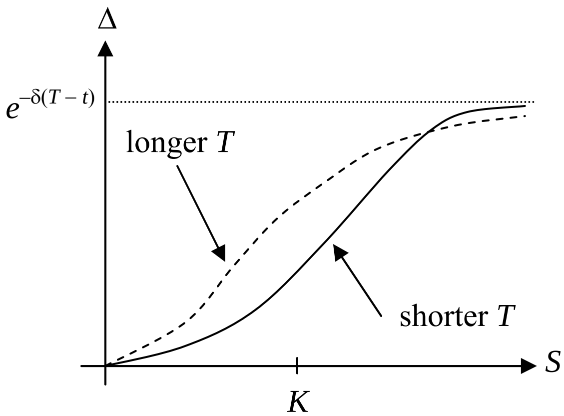
$$-0.3903 + (r - 0) \times 10 \times (-0.5520) + \frac{1}{2} \times 0.4^2 \times 10^2 \times (0.09889) = r \times 2.4954,$$

which gives $r = 0.05$.

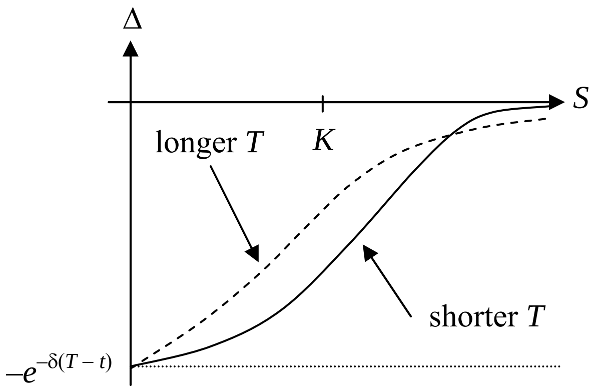
[END]

We now study the properties of the three Greeks for European calls and puts.

Properties of Δ



long call



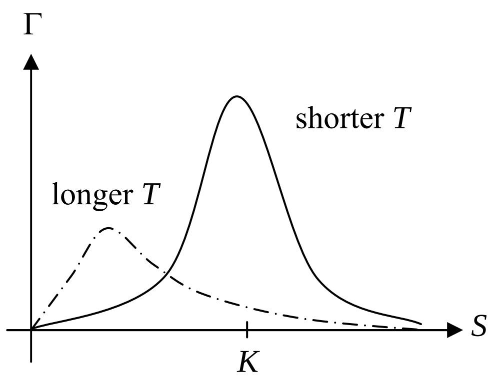
long put

- For calls, Δ is positive and bounded by 0 and $e^{-\delta(T-t)}$.
- For puts, Δ is negative and bounded by $-e^{-\delta(T-t)}$ and 0.
- If an option is deeply OTM (call: low S, put: high S), Δ would be close to 0.

  Explanation: When an option is very OTM, it is unlikely that it will be exercised and thus $V \approx 0$. In this case Δ would be close to 0 since it is not very sensitive to S (when S changes by a small amount, V is still very close to 0).
- If the option is deeply ITM, Δ approaches $e^{-\delta(T-t)}$ for calls and $-e^{-\delta(T-t)}$ for puts.

  Explanation: When a call is deeply ITM, we expect that the final payoff from the call would be $S(T) - K$, and hence $V \approx Se^{-\delta(T-t)} - Ke^{-r(T-t)}$, which means $\Delta \approx e^{-\delta(T-t)}$. The explanation for deeply ITM puts is similar.

Properties of Γ



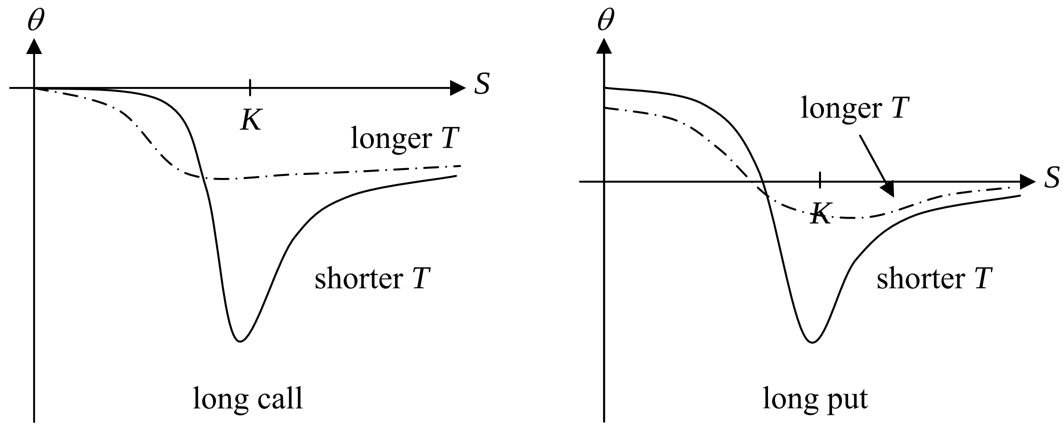
long call / put

– Calls and puts with the same strike and time to expiration have the same value of Γ.

– For long positions of calls and puts, Γ must be positive. Recall that a function is said to be convex if its second derivative is always non-negative. European calls and puts are hence called convex derivatives.

– Since Δ does not change much (it reaches either 0 or $\pm e^{-\delta(T-t)}$) when a call / put is deeply OTM or ITM, Γ is close to 0 when S is very low or very high.

Properties of θ



– The value of θ can be positive or negative. It is usually negative because call and put prices tend to drop as time passes.

One exception is a deeply in-the-money European put on a nondividend-paying stock. When a put is very ITM, we expect that the final payoff from the put would be $K - S(T)$, and hence $V \approx Ke^{-r(T-t)} - S$. This means $\theta \approx rKe^{-r(T-t)} > 0$.

Another exception is a deeply in-the-money European call on a currency with a very high interest rate.

– The theta of a European call on a nondividend-paying stock is always negative.

– The theta of a deeply OTM option is close to zero, while the theta of an at-the-money option is large and negative.

The Delta-Gamma-Theta Approximation

Apart from quantifying the risk of a derivative, delta, gamma and theta can also be used to approximate the price of a derivative when t or S changes by a small amount.

Suppose that at time t, the price of the derivative is $V(S, t)$. If the stock price suddenly changes to $S + \varepsilon$, how would the price of the derivative change? By Taylor's theorem, we have

$$V(S + \varepsilon, t) \approx V(S, t) + V_S(S, t) \times \varepsilon + \frac{1}{2}V_{SS}(S,t)\varepsilon^2 .$$

This leads to the following result.

FORMULA

Delta-Gamma Approximation

$$V(S + \varepsilon, t) \approx V(S, t) + \Delta(S, t) \times \varepsilon + \frac{1}{2}\Gamma(S,t)\varepsilon^2$$

If we drop the gamma term, the resulting formula

$$V(S + \varepsilon, t) \approx V(S, t) + \Delta(S, t) \times \varepsilon$$

is called a "delta approximation".

Example 3.2.4 [MFE 07 May #19]

Assume that the Black-Scholes framework holds. The price of a nondividend-paying stock is \$30.00. The price of a put option on this stock is \$4.00.

You are given $\Delta = -0.28$ and $\Gamma = 0.10$. Using the delta-gamma approximation, determine the price of the put option if the stock price changes to \$31.50.

(A) \$3.40 (B) \$3.50 (C) \$3.60 (D) \$3.70 (E) \$3.80

Solution

We have $V(S, t) = 4$, $S = 30$, $S + \varepsilon = 31.5$, and hence $\varepsilon = +1.5$.

Using a delta-gamma approximation, we have

$$V(S + \varepsilon, t) \approx 4 + (-0.28 \times 1.5) + 0.5(0.1 \times 1.5^2) = 3.6925$$

So the answer is (D).

[END]

It is unusual that S jumps "suddenly." A more comprehensive description is that the stock price changes from $S(t)$ to $S(t + h)$ when time proceeds from t to $t + h$. We can model this mathematically with a delta-gamma-theta approximation, which is derived from the multivariate version of Taylor's theorem.

FORMULA

Delta-Gamma-Theta Approximation

$$V(S(t+h), t+h) \approx V(S(t), t) + \Delta\varepsilon + \frac{1}{2}\Gamma\varepsilon^2 + \theta h$$

where $\varepsilon = S(t+h) - S(t)$, and the three Greeks are evaluated at $S(t)$ and t.

Delta, Gamma and Theta of a Portfolio of Derivatives

Suppose that an investor forms a portfolio with n derivatives written on the same underlying stock S. The investor takes a position of w_i units of the ith derivative, whose price is denoted by V_i. If $w_i > 0$, then it is a long position; and vice versa. The value of the portfolio is given by

$$P = \sum_{i=1}^{n} w_i V_i \,.$$

Hence, the delta of the portfolio is

$$\frac{\partial P}{\partial S} = \sum_{i=1}^{n} w_i \frac{\partial V_i}{\partial S} = \sum_{i=1}^{n} w_i \Delta_i \,.$$

The above says that the delta of the portfolio is the sum of the deltas of the individual portfolio components. This property also applies to the gamma and theta of the portfolio.

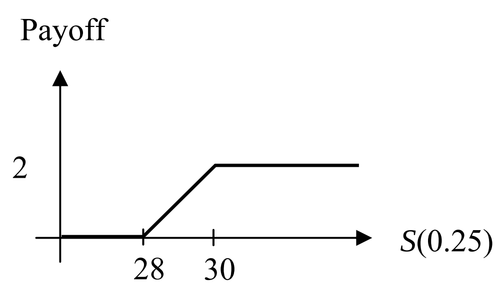
Example **3.2.5**

Suppose that S is a nondividend-paying stock which has a current price of 30. Assume the Black-Scholes framework, and that the volatility of the stock is 22%. The continuously compounded risk-free interest rate is 5%.

Compute the time-0 price and delta of the following two derivatives:

(a) a straddle that has a time-0.25 payoff of $| S(0.25) - 30 |$;

(b) a bull spread that has a time-0.25 payoff of $\max[0, \min(S(0.25), 30) - 28]$:



Solution

(a) The payoff of the straddle can be decomposed into

$$[S(0.25) - 30]_+ + [30 - S(0.25)]_+.$$

As a result, the straddle is just a 30-strike call plus a 30-strike put.

To find the price of the straddle, we calculate the following:

$$d_1 = \frac{\ln\frac{30}{30} + (0.05 + \frac{0.22^2}{2}) \times 0.25}{0.22\sqrt{0.25}} = 0.1686,\ d_2 = 0.1686 - 0.22\sqrt{0.25} = 0.0586$$

$N(d_1) = 0.5675$, $N(d_2) = 0.5239$

The call price is $30 \times 0.5675 - 30e^{-0.05/4} \times 0.5239 = 1.50324$.

The put price is $30e^{-0.05/4} \times (1 - 0.5239) - 30 \times (1 - 0.5675) = 1.13057$.

The price of the straddle is $1.50324 + 1.13057 = 2.6338$

We then calculate the deltas for the calls and puts:

– The call delta is $N(d_1) = 0.5675$.

– The put delta is $0.5675 - 1 = -0.4325$.

Thus, the delta of the straddle is $0.5675 + (-0.4325) = 0.1350$.

(b) The payoff of the bull spread can be decomposed into

$$[S(0.25) - 28]_+ - [S(0.25) - 30]_+.$$

As a result, the spread is just a 28-strike call minus a 30-strike call.

To find the price of 28-strike call,

$$d_1 = \frac{\ln\frac{30}{28} + (0.05 + \frac{0.22^2}{2}) \times 0.25}{0.22\sqrt{0.25}} = 0.7958,\ d_2 = 0.7958 - 0.22\sqrt{0.25} = 0.6858$$

$N(d_1) = 0.7881$, $N(d_2) = 0.7549$

The 28-strike call price is $30 \times 0.7881 - 28e^{-0.05/4} \times 0.6858 = 2.76837$.

The price of the bull spread is $2.76837 - 1.50324 = 1.2651$.

Note that the delta of the 28-strike call is $N(d_1) = 0.7881$.

Thus, the delta of the bull spread is $0.7881 - 0.5675 = 0.2206$.

[END]