MODELS FOR QUANTIFYING RISK

SECOND EDITION

ROBIN J. CUNNINGHAM, FSA, PH.D.
THOMAS N. HERZOG, ASA, PH.D.
RICHARD L. LONDON, FSA

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PREFACE

The analysis and management of financial risk is the fundamental subject matter of the discipline of actuarial science, and is therefore the basic work of the actuary. In order to manage financial risk, by use of insurance schemes or any other risk management technique, the actuary must first have a framework for quantifying the magnitude of the risk itself. This is achieved by using mathematical models that are appropriate for each particular type of risk under consideration. Since risk is, almost by definition, probabilistic, it follows that the appropriate models will also be probabilistic, or stochastic, in nature.

This new textbook, appropriately entitled *Models for Quantifying Risk*, addresses the major types of financial risk analyzed by actuaries, and presents a variety of stochastic models for the actuary to use in undertaking this analysis. It is designed to be appropriate for a two- or three-semester university course in basic actuarial science for third-year or fourth-year undergraduate students or entry-level graduate students. It is also intended to be an appropriate text for use by candidates in preparing for Exam M of the Society of Actuaries or Exam 3 of the Casualty Actuarial Society.

One way to manage financial risk is to *insure* it, which basically means that a second party, generally an insurance company, is paid a fee to assume the risk from the party initially facing it. Historically the work of actuaries was largely confined to the management of risk within an insurance context, so much so, in fact, that actuaries were thought of as “insurance mathematicians” and actuarial science was thought of as “insurance math.” Although the insurance context remains a primary environment for the actuarial management of risk, it is by no means any longer the only one.

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However, in recognition of the insurance context as the original setting for actuarial analysis and management of financial risk, we have chosen to make liberal use of insurance terminology and notation to describe many of the risk quantification models presented in this text. The reader should always keep in mind, however, that this frequent reference to an insurance context does not reduce the applicability of the models to risk management situations in which no use of insurance is involved.

The text is written in a manner that assumes each reader has a strong background in calculus, linear algebra, the theory of compound interest, and probability. (A familiarity with statistics is not presumed.) Within the context of using the text for SOA or CAS exam preparation, the expectation is that the reader has previously passed the preliminary exams on probability and financial mathematics, or has at least done a serious preparation for them.

The original edition of the text was designed to completely cover the topics expected to be included on the new SOA Exam M, which was first offered in May 2005. As it turned out, several topics in the original edition were never included on Exam M from the beginning, and several others have since been deleted from the Exam M syllabus.

The new edition is organized into three sections. The first, consisting of Chapters 1 and 2, presents a review of interest theory and probability, respectively. The content of these chapters is very much needed as background to later material. They are included in the text for readers needing a comprehensive review of the topics. For those requiring an original textbook on either of these topics, we recommend the works by either Broverman [5] or Kellison [16] for interest theory, and the work by Hassett and Stewart [12] for probability.

The second section, made up of Chapters 3-10, addresses the topic of survival-contingent payment models, traditionally referred to as *life contingencies*, and the third section, consisting of Chapters 11-14, deals with the topic of aggregate payment models, traditionally referred to as *risk theory*. Only the life contingencies material is included on Exam M, effective with the May 2007 exam. (The risk theory material is covered on Exam C; candidates for that exam might find Chapters 11-14 of this text to be very useful at that time.)

The general topic of stochastic processes is no longer included on Exam M, but the specific topic of discrete-time Markov Chains is important background material for understanding the multi-state models presented in Section 10.7. Students who have not had a university course in this background material can review it by studying Appendix A in this text. Candidates for SOA Exam M or CAS Exam 3 will find an excellent complement to the material in Section 10.7 in a study note by J.W. Daniel entitled "Multi-State Transition Models with Actuarial Applications."

Another specific type of stochastic process, namely the Poisson process (including the compound Poisson process) is included on the Exam M syllabus. Beginning with the basic Poisson probability distribution, the Poisson process topic is covered in this text in Sections 11.3 and 14.1.

Certain actuarial risk models are most efficiently evaluated through simulation, rather than by use of closed form analytic solutions. Appendix C illustrates the application of simulation to selected models. These illustrations will be easily understood by readers who have had a basic course in stochastic simulation; for those who have not, or who require a review of the topic, we have included a summary of the practice of simulation in Appendix B.

The writing team would like to thank a number of people for their contributions to the development of this text.

An early draft of the manuscript was thoroughly reviewed by Bryan V. Hearsey, ASA, of Lebanon Valley College and by Esther Portnoy, FSA, of University of Illinois. Portions of the manuscript were also reviewed by Warren R. Luckner, FSA, and his graduate student Luis Gutierrez at University of Nebraska-Lincoln. Kristen S. Moore, ASA, used the earlier draft as a supplemental text in her courses at University of Michigan. Thorough reviews of the original edition were also conducted by James W. Daniel, ASA, of University of Texas, Professor Jacques Labelle, Ph.D., of Université du Québec à Montréal, and a committee appointed by the Society of Actuaries. The revised sections in this Second Edition were also reviewed by Professors Daniel and Hearsey. All of these academic colleagues made a number of useful comments that have contributed to an improved published text.

Special thanks goes to the students enrolled in Math 287-288 at University of Connecticut during the 2004-05 academic year, where the original text was classroom-tested, and to graduate student Xiumei Song, who developed the computer technology material presented in Appendix D.

Thanks also to the folks at ACTEX Publications, particularly Gail A. Hall, FSA, the project editor, Marilyn J. Baleshiski, who did the typesetting and graphic arts, and Kathleen H. Borkowski, who designed the text's cover.

Finally, a very special acknowledgment is in order. When the Society of Actuaries published its textbook *Actuarial Mathematics* in the mid-1980s, Professor Geoffrey Crofts, FSA, then at University of Hartford, made the observation that the authors' use of the generic symbol Z as the present value random variable for *all* insurance models and the generic symbol Y as the present value random variable for *all* annuity models was confusing. He suggested that the present value random variable symbols be expanded to identify more characteristics of the models to which each related, following the principle that the present value random variable be notated in a manner consistent with the standard International Actuarial Notation used for its expected value. Thus one should use, for example, $\bar{Z}_{x:\overline{n}|}$ in the case of the continuous endowment insurance model and ${}_{n|}\ddot{Y}_x$ in the case of the n-year deferred annuity-due model, whose expected values are denoted $\bar{A}_{x:\overline{n}|}$ and ${}_{n|}\ddot{a}_x$, respectively. Professor Crofts' notation has been adopted throughout our textbook, and we wish to thank him for suggesting this very useful idea to us.

We wish you good luck with your studies and your exam preparation.

Robin J. Cunningham, FSA
San Rafael, California

Winsted, Connecticut
August 2006

Thomas N. Herzog, ASA
Reston, Virginia

Richard L. London, FSA
Storrs, Connecticut

CHAPTER THREE

SURVIVAL MODELS (CONTINUOUS PARAMETRIC CONTEXT)

A *survival model* is simply a probability distribution for a particular type of random variable. Thus the general theory of probability, as reviewed in Chapter 2, is fully applicable here. However the particular history of the survival model random variable is such that specific terminology and notation has developed, particularly in an actuarial context. In this chapter (and the next) the reader will see this specialized terminology and notation, and recognize that it is *only* the terminology and notation that is new; the underlying probability theory is the same as that applying to any other continuous random variable and its distribution.

In actuarial science, the survival distribution is frequently summarized in tabular form, which is called a *life table*.[1] Because the life table form is so prevalent in actuarial work, we will devote a full chapter to it in this textbook (see Chapter 4).

3.1 THE AGE-AT-FAILURE RANDOM VARIABLE

We begin our study of survival distributions by defining the generic concept of *failure*. In any situation involving a survival model, there will be a defined entity and an associated concept of *survival*, and hence of failure, of that entity. Here are some examples of entities and their associated random variables.

(1) The operating lifetime of a light bulb. The bulb is said to survive as long as it keeps burning, and fails at the instant it burns out.

(2) The duration of labor/management harmony. The state of harmony

[1] Alternatively, the tabular model is also called a *mortality table*.

continues to survive as long as regular work schedules are met, and fails at the time a strike is called. (Conversely, we could model the duration of a strike, where the strike survives until it is settled and workers return to the job. The settlement event constitutes the failure of the strike status.)

(3) The lifetime of a new-born person. The person survives until death occurs, which constitutes the failure of the human entity.

Let X denote the continuous random variable for the age of the entity at the instant it fails. We assume that the entity exists at age 0, so the domain of the random variable X is $X > 0$. We refer to X as the *age-at-failure* random variable. We will consider the terms "failure" and "death" to be synonymous, so we will also refer to X as the *age-at-death* random variable.[2]

It is easy to see that the numerical value of the age at failure is the same as the *length of time* that survival lasts until failure occurs, since the variable begins at age 0. This fundamental point is illustrated in Figure 3.1.

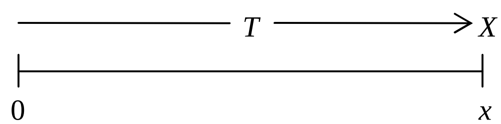

FIGURE 3.1

Let T denote the continuous random variable for the length of time from age 0 until failure occurs. We refer to T as the *time-to-failure* random variable. If failure occurs at exact age x, then clearly we have $X = x$ and $T = x$ as well.

Although we could use the age-at-failure and the time-to-failure variables interchangeably, we will consistently use X, the age-at-failure random variable, in all cases where survival is measured from age 0. Later (see Section 3.3) we will consider the case where the entity of interest is

[2] In practice, age-at-failure is often used for inanimate objects, such as light bulbs or labor strikes, and age-at-death is used for animate entities, such as laboratory animals or human persons under an insurance arrangement.

known to have survived to some age $x > 0$. Then the time-to-failure random variable T will not be identical to the age-at-failure random variable X, although they will be related to each other by $X = x + T$. When dealing with this more general case we will do our thinking in terms of the time-to-failure random variable.

3.1.1 THE CUMULATIVE DISTRIBUTION FUNCTION OF *X*

For the age-at-failure random variable X, we denote its CDF by the usual

$$F_X(x) = Pr(X \leq x), \tag{3.1}$$

for $x \geq 0$. We have already noted, however, that $X = 0$ is not possible, so we will always consider that $F_X(0) = 0$. We observe that $F_X(x)$ gives the probability that failure will occur prior to (or at) precise age x for our entity known to exist at age 0. In actuarial notation, this probability is denoted by ${}_xq_0$, so we have

$${}_xq_0 = F_X(x) = Pr(X \leq x). \tag{3.2}$$

3.1.2 THE SURVIVAL DISTRIBUTION FUNCTION OF *X*

The *survival distribution function* (SDF) for the survival random variable X is denoted by $S_X(x)$, and is defined by

$$S_X(x) = 1 - F_X(x) = Pr(X > x), \tag{3.3}$$

for $x \geq 0$. Since we take $F_X(0) = 0$, it follows that we will always take $S_X(0) = 1$. The SDF gives the probability that the age at failure exceeds x, which is the same as the probability that the entity known to exist at age 0 will survive to age x. Since the notion of infinite survival is unrealistic, we consider that

$$\lim_{x \to \infty} S_X(x) = 0 \tag{3.4a}$$

and

$$\lim_{x \to \infty} F_X(x) = 1. \tag{3.4b}$$

In actuarial notation, the probability represented by $S_X(x)$ is denoted ${}_x p_0$, so we have

$$ {}_x p_0 \quad = \quad S_X(x) \quad = \quad Pr(X > x). \tag{3.5} $$

In probability textbooks in general, the CDF is given greater emphasis than is the SDF. (Some textbooks do not even define the SDF at all.) But when we are dealing with an age-at-failure random variable, and its associated distribution, the SDF will receive greater attention.

EXAMPLE 3.1

Use both the CDF and the SDF to express the probability that an entity known to exist at age 0 will fail between the ages of 10 and 20.

SOLUTION

We seek the probability that *X* will take on a value between 10 and 20. In terms of the CDF we have

$$ Pr(10 < X \le 20) \quad = \quad F_X(20) - F_X(10). $$

Since $S_X(x) = 1 - F_X(x)$, then we also have

$$ Pr(10 < X \le 20) \quad = \quad S_X(10) - S_X(20). $$ □

3.1.3 THE PROBABILITY DENSITY FUNCTION OF *X*

For a continuous random variable in general, the *probability density function* (PDF), denoted $f_X(x)$, is defined as the derivative of $F_X(x)$. Thus we have

$$ f_X(x) \quad = \quad \frac{d}{dx} F_X(x) \quad = \quad -\frac{d}{dx} S_X(x), \tag{3.6} $$

for $x > 0$. Consequently,

$$F_X(x) = \int_0^x f_X(y)\,dy \tag{3.7}$$

and

$$S_X(x) = \int_x^\infty f_X(y)\,dy. \tag{3.8}$$

Of course it must be true that

$$\int_0^\infty f_X(y)\,dy = 1. \tag{3.9}$$

Although we have given mathematical definitions of $f_X(x)$, it will be useful to describe $f_X(x)$ more fully in the context of the age-at-failure random variable. Whereas $F_X(x)$ and $S_X(x)$ are probabilities which relate to certain *time intervals*, $f_X(x)$ relates to a *point of time*, and is not a probability. It is the density of failure *at* age *x*, and is therefore an *instantaneous* measure, as opposed to an interval measure.

It is important to recognize that $f_X(x)$ is the *unconditional* density of failure at age *x*. By this we mean that it is the density of failure at age *x* given *only* that the entity existed at $x = 0$. The significance of this point will become clearer in the next subsection.

3.1.4 THE HAZARD RATE FUNCTION OF *X*

Recall that the PDF of X, $f_X(x)$, is the *unconditional* density of failure at age *x*. We now define a *conditional* density of failure at age *x*, with such density conditional on survival to age *x*. This conditional instantaneous measure of failure at age *x*, given survival to age *x*, is called the *hazard rate* at age *x*, or the *hazard rate function* (HRF) when viewed as a function of *x*. (In some textbooks the hazard rate is called the *failure rate*.) It will be denoted by $\lambda_X(x)$.

In general, if a conditional measure is multiplied by the probability of obtaining the conditioning event, then the corresponding unconditional measure will result. Specifically,

(Conditional density of failure at age *x*, given survival to age *x*)
$\times$ (Probability of survival to age *x*)
= (Unconditional density of failure at age *x*).

Symbolically this states that

$$\lambda_X(x) \cdot S_X(x) = f_X(x), \tag{3.10}$$

or

$$\lambda_X(x) = \frac{f_X(x)}{S_X(x)}. \tag{3.11}$$

Equations (3.11) and (3.6) give formal definitions of the HRF and the PDF, respectively, of the age-at-failure random variable. Along with the definitions it is also important to have a clear understanding of the *descriptive meanings* of $\lambda_X(x)$ and $f_X(x)$. They are both instantaneous measures of the density of failure at age *x*; they differ from each other in that $\lambda_X(x)$ is conditional on survival to age *x*, whereas $f_X(x)$ is unconditional (i.e., given only existence at age 0).

In the actuarial context of survival models for animate objects, including human persons, failure means death, or mortality, and the hazard rate is normally called the *force of mortality*. We will discuss the actuarial context further in Section 3.1.6 and in Chapter 4.

Some important mathematical consequences follow directly from Equation (3.11). Since $f_X(x) = -\frac{d}{dx} S_X(x)$, it follows that

$$\lambda_X(x) = \frac{-\frac{d}{dx} S_X(x)}{S_X(x)} = -\frac{d}{dx} \ln S_X(x). \tag{3.12}$$

Integrating, we have

$$\int_0^x \lambda_X(y)\, dy = -\ln S_X(x), \tag{3.13}$$

or

$$S_X(x) = \exp\left[-\int_0^x \lambda_X(y)\, dy\right]. \tag{3.14}$$

The *cumulative hazard function* (CHF) is defined to be

$$\Lambda_X(x) \quad = \quad \int_0^x \lambda_X(y)\, dy \quad = \quad -\ln S_X(x), \tag{3.15}$$

so that

$$S_X(x) \quad = \quad e^{-\Lambda_X(x)}. \tag{3.16}$$

EXAMPLE 3.2

An age-at-failure random variable has a distribution defined by

$$F_X(x) \quad = \quad 1-.10(100-x)^{1/2},$$

for $0 \le x \le 100$. Find (a) the PDF and (b) the HRF for this random variable.

SOLUTION

(a) The PDF is given by

$$\begin{aligned} f_X(x) \quad = \quad \frac{d}{dx}F_X(x) \quad &= \quad -(.10)(.50)(100-x)^{-1/2}\cdot(-1) \\ &= \quad .05(100-x)^{-1/2}. \end{aligned}$$

(b) The HRF is given by

$$\lambda_X(x) \quad = \quad \frac{f_X(x)}{S_X(x)} \quad = \quad \frac{.05(100-x)^{-1/2}}{.10(100-x)^{1/2}} \quad = \quad .50(100-x)^{-1}. \qquad \square$$

3.1.5 THE MOMENTS OF THE AGE-AT-FAILURE RANDOM VARIABLE *X*

The first moment of a continuous random variable defined on $[0,\infty)$ is given by

$$E[X] \quad = \quad \int_0^\infty x\cdot f_X(x)\, dx, \tag{3.17}$$

if the integral exists, and otherwise the first moment is undefined. Integration by parts yields the alternative formula

$$E[X] = \int_0^\infty S_X(x)\,dx, \tag{3.18}$$

a form which is frequently used to find the first moment of an age-at-failure random variable.

The second moment of X is given by

$$E[X^2] = \int_0^\infty x^2 \cdot f_X(x)\;dx, \tag{3.19}$$

if the integral exists, so the variance of X can be found from

$$Var(X) = E[X^2] - \{E[X]\}^2. \tag{3.20}$$

Specific expressions can be developed for the moments of X for specific forms of $f_X(x)$. This will be pursued in the following section.

Another property of the age-at-failure random variable that is of interest is its *median* value. We recall that the median of a continuous random variable is the value for which there is a 50% chance that X will exceed (and thus also not exceed) that value. Mathematically, y is the median of X if

$$Pr(X > y) = Pr(X < y) = \frac{1}{2}, \tag{3.21}$$

so that $S_X(y) = F_X(y) = \frac{1}{2}$.

3.1.6 ACTUARIAL SURVIVAL MODELS

When the age-at-failure random variable is considered in an actuarial context, special symbols are used for some of the concepts defined in this section. The hazard rate, now called the force of mortality, is denoted by μ_x, rather than $\lambda_X(x)$. Thus we have

$$\mu_x = \frac{-\frac{d}{dx} S_X(x)}{S_X(x)} = -\frac{d}{dx} \ln S_X(x). \tag{3.22}$$

It is customary to denote the first moment of *X* by $\overset{\circ}{e}_0$. Thus we have

$$\overset{\circ}{e}_0 = E[X] = \int_0^\infty x \cdot f_X(x)\, dx. \tag{3.23}$$

Since $\overset{\circ}{e}_0$ is the unconditional expected value of *X*, given only alive at $x = 0$, it is called the *complete expectation of life at birth*.[3]

We recognize that the moments of *X* given above are all unconditional. Conditional moments, and other conditional measures, are defined in Section 3.3, and the standard actuarial notation for them is reviewed in Chapter 4.

EXAMPLE 3.3

For the distribution of Example 3.2, find (a) *E*[*X*] and (b) the median of the distribution.

SOLUTION

(a) The expected value is given by Equation (3.18) as

$$\begin{aligned} E[X] &= \int_0^{100} .10(100-x)^{1/2}\, dx \\ &= -\left(\frac{2}{3}\right)(.10)(100-x)^{3/2}\Big|_0^{100} \\ &= \left(\frac{2}{3}\right)(.10)(100)^{3/2} = \frac{200}{3}. \end{aligned}$$

[3] The significance of the adjective "complete" will become clearer when we consider an alternative measure of the expectation of life in Sections 3.3.6 and 4.3.4.

(b) The median is the value of y satisfying $S_X(y) = .10(100-y)^{1/2} = .50$, which solves for $y = 75$. □

CHAPTER FIVE

CONTINGENT PAYMENT MODELS (INSURANCE MODELS)

In this chapter we address the concept of models for a single payment arising from the occurrence of a defined random event. This description, and the mathematics that follows, is intended to be very general.

A particular random event is defined. If, and when, that event occurs, a single payment of predetermined amount is paid as a consequence of the occurrence of the event. A wide variety of examples can be cited, including the following:

(1) I will pay you $10.00 the next time your favorite football team wins a game.
(2) The outstanding balance of a loan becomes payable if the borrower defaults on the loan.
(3) The face amount of a life insurance policy becomes payable upon the death of the person insured under the policy.

Note what is common to all three examples: a payment is made due to the occurrence of a defined random event. The payments are *contingent* on the occurrence of the associated events. Models representing such payments are collectively referred to as *contingent payment models*. In those cases where a financial loss results from the occurrence of the event, and the loss is reimbursed (in whole or in part) by another party, then we say the loss is *insured*. The party reimbursing the loss is an *insurer*, and the model describing the reimbursement arrangement is an *insurance model*.

Note that the mathematics of a contingent payment model does not depend on whether or not the loss is insured. Thus we use *contingent payment model* as the more general concept, and *insurance model* as a special case. The meaning of this will become clearer as the mathematics unfolds throughout the chapter.

5.1 DISCRETE STOCHASTIC MODELS

A common feature of the contingent payment models presented in this text is that the associated random event occurs at some *point in time* (if indeed it occurs at all). Furthermore, in many actuarial applications, the random event of interest is the *failure* of some defined *status* to continue to exist. In the examples presented at the beginning of this chapter, (1) the event of winning a game represents the "failure" of the continuation of a losing streak, (2) the event of default represents the failure of the loan to continue to be in good standing, and (3) the event of death represents the failure of the continued survival of the person insured under the policy. In this section of the chapter we develop the mathematics of contingent payment models wherein the time of failure of the status of interest is observed to occur in some finite time interval.

5.1.1 THE DISCRETE RANDOM VARIABLE FOR TIME OF FAILURE

Recall the discrete random variable K_x defined in Section 3.3.6 to denote the *time interval of failure* for a status of interest. We think of x as an *identifying characteristic* of the status of interest as of time 0. A common example of this will be that x denotes the age of the status at that time. Thus, in the life insurance example, K_x will denote the time interval of failure for a person (the "status") who is age x at time 0, the time at which the insurance is issued. This is further pursued in Section 5.1.4.

5.1.2 THE PRESENT VALUE RANDOM VARIABLE

Suppose a unit of money is payable at the *end* of the time interval in which failure occurs. Then if $K_x = k$, so that failure occurs in the interval $(k-1, k]$, a unit is paid at time k. Assuming a constant rate of compound interest throughout the model, the present value at time 0 of this payment is denoted by v^k. But the time of payment is a random variable, so the present value of payment is likewise a random variable, which we denote by Z_x. Thus we have

$$Z_x = v^{K_x}, \tag{5.1}$$

for $K_x = 1, 2, \ldots$.[1]

The expected value (or first moment) of the random variable Z_x is denoted by A_x. Thus we have

$$A_x \quad = \quad E[Z_x] \quad = \quad \sum_{k=1}^{\infty} v^k \cdot Pr(K_x = k), \tag{5.2}$$

a case of finding the expected value of a function of the discrete random variable K_x. (See Equation 2.3 in Section 2.1.1.) The second moment of Z_x is denoted by ${}^2A_x$, and is given by

$${}^2A_x \quad = \quad E[Z_x{}^2] \quad = \quad \sum_{k=1}^{\infty} (v^k)^2 \cdot Pr(K_x = k). \tag{5.3}$$

Recall from the theory of compound interest (see Section 1.1) that the discount factor v is related to the force of interest δ by $v = e^{-\delta}$. If we let $v' = v^2$, then it follows that $v' = (e^{-\delta})^2 = e^{-2\delta}$. In Equation (5.3) we can substitute $(v^k)^2 = v^{2k} = (v^2)^k = (v')^k$, so that Equation (5.3) becomes

$${}^2A_x \quad = \quad E[Z_x{}^2] \quad = \quad \sum_{k=1}^{\infty} (v')^k \cdot Pr(K_x = k). \tag{5.4}$$

This shows that ${}^2A_x$ is the same kind of function as is A_x, except that it is calculated at a force of interest that is double the force of interest used to calculate A_x. Recall also from compound interest theory that if $\delta' = 2\delta$, then $1 + i' = e^{\delta'} = e^{2\delta} = (e^{\delta})^2 = (1+i)^2$, so that $i' = (1+i)^2 - 1$. Thus if interest rate i is used to calculate A_x, then ${}^2A_x$ is calculated at rate $i' = (1+i)^2 - 1 = 2i + i^2$, but not $i' = 2i$. It is important to remember

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[1] Some texts (see, for example, Bowers et al. [4]) denote the time interval of failure by the curtate duration random variable $K(x)$, defined in Section 3.3.6. In that case, the present value random variable would be $Z_x = v^{K(x)+1}$, since $K(x)$ denotes the duration at the beginning of the failure interval but the payment is made at the end.

that ${}^2A_x$ is calculated at double the *force of interest*, not double the *effective rate of interest*.

The variance of the present value random variable Z_x is then given by

$$Var(Z_x) \quad = \quad {}^2A_x - A_x^{\,2}. \tag{5.5}$$

Because K_x is discrete, then $Z_x = v^{K_x}$ is also discrete, so the full distribution of Z_x can be tabulated from the distribution of K_x. Having the full distribution enables us to find the median (or any other percentile) of the random variable Z_x. This is illustrated in part (c) of the following example.

EXAMPLE 5.1

A payment of $10.00 will be made at the end of the week during which a family's supply of laundry detergent runs out. The family's usage of detergent is variable, so the week of exhaustion of the supply is a random variable *K*, with the following distribution:

k	***Pr(K=k)*** [2]
1	.20
2	.30
3	.20
4	.15
5	.15

Let $Z = 10v^K$ denote the present value of payment random variable. Find (a) the mean, (b) the variance, and (c) the median of *Z*, using an interest rate of $i = .01,$ effective per week.

SOLUTION

(a) By Equation (5.2) the mean of *Z* is

[2] It is not possible for the supply to last more than five weeks.

$$E[Z] = \sum_{k=1}^{5} 10v^k \cdot Pr(K=k)$$
$$= 10\left[\frac{.20}{1.01}+\frac{.30}{(1.01)^2}+\frac{.20}{(1.01)^3}+\frac{.15}{(1.01)^4}+\frac{.15}{(1.01)^5}\right] = 9.73094.$$

(b) First we find $i' = (1.01)^2 - 1 = .0201.$ Then from Equation (5.3) we have

$$E[Z^2] = \sum_{k=1}^{5} \left(10v^k\right)^2 \cdot Pr(K=k)$$
$$= 100\sum_{k=1}^{5} (v')^k \cdot Pr(K=k)$$
$$= 100\left[\frac{.20}{1.0201}+\frac{.30}{(1.0201)^2}+\frac{.20}{(1.0201)^3}+\frac{.15}{(1.0201)^4}+\frac{.15}{(1.0201)^5}\right]$$
$$= 94.70782.$$

Then $Var(Z) = 94.70782 - (9.73094)^2 = .01663.$

(c) The five possible values of $Z = 10v^K$ follow from the five possible values of *K* itself. The complete distribution of *Z* is as follows:

k	$z = 10v^k$	$Pr(Z=z)$
5	9.51466	.15
4	9.60980	.15
3	9.70591	.20
2	9.80296	.30
1	9.90099	.20

The median of *Z* is the value *m* for which

$$Pr(Z \le m) = .50. \qquad (5.6)$$

In this case the median is $z = 9.70591,$ since $Pr(Z \le 9.70591) = .50.$ (Note that the median of a discrete random variable is not always as clearly apparent as it is in this case. See Footnote 3 on page 23.) □