Financial Mathematics: Derivatives Markets

Course notes for SOA Exam FM and CAS Exam 2

- Meets learning objectives of the Society of Actuaries Exam FM and Casualty Actuarial Society Exam 2
- Over 50 worked examples and practice questions
- Clear explanations and guidance from experienced instructors
- Free online solutions manual
Introduction

These Exam FM notes are designed to be a replacement for the material on the Society of Actuaries syllabus from Derivatives Markets. Each study session corresponds to the study sessions in our Study Program Guide and includes practice questions at the end of each session. The solutions for these practice questions can be found online at wwwbpptraining.com.

BPP’s Question & Answer bank (available separately) is divided into the corresponding study sessions. You’ll find full solutions to every question at the end of the Question & Answer Bank.

Good luck in the exam.

BPP Exam FM Tutor Team

November 2011
Study Session 9

Introduction to derivatives

- This session’s course notes cover a relatively straightforward introduction to derivatives, but don’t worry, it will get much more complicated soon enough!

- Our policy on rounding is to keep full accuracy within intermediate calculations, even though an intermediate result may be shown as a rounded value.
9.1 Derivative security overview

A derivative security is a financial instrument whose value is derived from the return or price of another asset, which is called the underlying asset. In practice, derivatives are based upon a wide range of underlying assets, including:

- financial securities
- commodities, such as crude oil, gold and coffee beans
- indexes, based on security prices and commodity prices.

More recently, derivatives have been developed that derive their value from an underlying liability or even an underlying economic index, such as employment statistics. In fact, in principle, a derivative can be agreed based on almost any underlying asset.

In the study sessions that follow, we will consider such derivatives as forwards, futures, options, and swaps, as well as various combinations of these derivatives.

How derivatives are used

In general, derivatives are used to manage exposure to risk and returns, by a wide range of investors and also by the producers of commodities. For example, the producer of a product is exposed to the risk that the price of the product falls before they are able to bring it to market. As we shall see in Study Session 12, this risk could be reduced using a forward contract.

Likewise, investors are exposed to many risks, including the risk that asset prices will change over time. The potential for an asset’s price to rise and fall is sometimes called price volatility risk and can be measured by the standard deviation of an asset’s price over time.

While investors may be content to be exposed to price volatility risk, they often prefer to manage risk according to their own unique characteristics. Derivatives are a tool that can be used in the risk-management process either to increase risk or mitigate risk depending upon the investor’s preference. This theme of risk management appears throughout the derivatives material.

As far as we are concerned in this course, there are five primary uses of derivatives:

1. Derivatives can be used for speculation to increase exposure to risk and hence investment returns. We’ll discuss this in Study Session 11. Derivatives also enable the potential gain (or loss) from speculation to be leveraged (ie magnified) relative to the initial investment.

2. Derivatives can be used as a hedge to reduce exposure to risk. We’ll discuss this in detail in Study Sessions 12, 13 and 14.

3. Derivatives can be used to reduce transaction costs, such as commissions and other trading costs, compared to trading in the underlying assets themselves.

4. Derivatives can be mis-priced with regard to the underlying assets. Investors may therefore be able to use derivatives to exploit such pricing anomalies and make arbitrage profits. We’ll discuss this in Study Session 13.

5. Finally, derivatives can be used to take advantage of differences in how tax laws, accounting rules, and other regulations may apply differently to derivatives compared to the underlying assets. Investors may therefore use derivatives to take advantage of this fact using a process called regulatory arbitrage. We’ll see an example of this in Chapter 12.
More generally, by offering an alternative to simply buying or selling the underlying asset itself, derivatives increase the range of possibilities to investors. In particular, derivatives can be used to create financial products that generate an investor’s required set of payoffs in order to satisfy that investor’s desired risk exposure. This process is called financial engineering.

**Derivative perspectives**

In addition to considering their uses, we can also consider derivatives from three different user perspectives:

1. the *end-user*, typically an investor (*eg* individuals, insurance companies, mutual funds, hedge funds), but sometimes the producer of a product, employs a derivative in the ways described above, *eg* to gain a desired risk exposure.

2. the *market-maker*, through whom end-users trade derivatives, and who therefore serves as an intermediary between different end users. Market-makers charge a fee for this service, which we’ll consider in more detail soon. The market-maker may want to take an offsetting position (*ie* a hedged position) to manage his or her own risk exposure.

3. the *economic observer*, such as a regulator or an economist, who observes derivative transactions, how derivatives are used, the operation of derivative markets *etc* and may even set rules for the transaction.

### 9.2 Financial markets

Derivatives play an increasingly important role in financial markets. Financial markets connect those who are willing to assume a certain type of risk with those who are willing to sell it. More specifically, financial markets:

- *facilitate transactions* between investors, who wish buy and sell exposure to different risks
- allow companies to *raise capital*, via the issuance of stocks and bonds
- allow companies to use derivatives to *hedge or insure* against risk exposure
- enable investors to *lower transactions costs* (*eg* with low-cost index mutual funds)
- permit the *diversification* of risks, to reduce exposure to price volatility risk
- allow *risk pooling* with others willing to share those risks, which enables insurance costs to be reduced
- *increase efficiency*, by allowing undesired risks to be sold to parties who are more willing to bear them.

The role of risk sharing is very important in the context of insurance. Insurance companies assume risk, pooling the risks from a significant number of policyholders. Insurance companies can hedge this risk with *reinsurance* to cede some of their liabilities to a reinsurance company willing to bear it. A reinsurance company can in turn share some of its risk with other parties, *eg* by selling a *catastrophe bond* that does not need to be repaid in full in the event of a specified catastrophe, such as a hurricane.
**Diversifiable vs. nondiversifiable risk**

Portfolio theory divides risk into two categories, namely diversifiable and nondiversifiable.

- **Diversifiable risk** is company-specific risk that can be eliminated by risk sharing via diversification within a financial market.
- **Nondiversifiable risk** is exposure to a market risk that cannot be diversified away.

Financial markets enable investors to eliminate their exposure to diversifiable risk, by sharing that risk with many others. At the same time, they also enable nondiversifiable risk to be passed on to those most willing to bear it.

### 9.3 The practicalities of using derivatives

The use of derivatives by companies and investors to manage their risks has increased hugely over recent years, although data to measure the extent of derivative usage have either not been very precise or readily available.

Derivatives may be traded on an exchange or they may be traded over-the-counter. As the name implies, exchanged-traded derivatives are traded on an exchange, such as the Chicago Board of Trade (CBOT). Exchanges standardize derivatives in order to facilitate trading and are highly regulated. Such derivatives are therefore quick, cheap and easy to trade. Over-the-counter (OTC) derivatives are not traded on an exchange, but are bought and sold through dealers and investment banks. OTC derivatives must be individually negotiated, with the advantage that they can be customized to meet the needs of a particular investor. The OTC market is not as regulated as the exchange market and offers more confidentiality among participants.

### 9.4 Buying and short selling financial assets

Buying and selling both an asset and a derivative involves transaction costs. There are two main types of transaction costs: the broker’s commission and the bid-ask spread.

The **commission** is simply a transaction fee, which can be expressed as either a percentage of the price or as a flat fee, eg as 0.1% of the total price paid or as $20 per transaction.

The **bid-ask spread** is the difference between the **bid price**, at which the broker will buy the asset, and the higher **ask price**, at which the broker sells an asset. Note that these terms are named from the broker’s perspective. In other words, the bid price is the buying price (ie the bidding price) the broker pays for a share of stock, which is the selling price the investor receives. The ask price is also known as the **offer price**, which is the price the broker offers to sell the share of stock, so the ask price is the buying price the investor pays. As such, the bid price is less than the ask (offer) price, and the broker profits from the bid-ask spread.
Buying an asset

When an investor buys an asset, it is referred to as a long position in the asset, and the investor usually has an opinion that the asset price will increase. The buyer therefore hopes to profit from selling the asset later at a higher price and is said to be bullish on the asset’s future prospects. The old adage, “buy low and sell high” applies, but to be more precise, let’s say “buy low now and sell high later” so that we can compare a long position with a short position. Likewise, an investor or producer who already owns an asset or product is also said to have a long position in the asset and will again profit from a price rise.

Finally, note that buying an asset is like lending cash since the buyer pays cash today to buy the asset and receives cash in the future when the asset is ultimately sold.

Short selling an asset

A short sale of an asset involves the sale of an asset now that is not currently owned by the seller and the later repurchase and return of the asset to its original owner.

A short sale is referred to as a short position in the asset, and the short seller usually believes that the price of the asset will fall. The short seller may be speculating by hoping to profit by buying back the asset in the future at a lower price than the short seller initially sold it at. The short seller is said to be bearish on the asset’s future prospects. A short sale can be described as “sell high now, buy back low later.”

Example

For example, suppose that shares in XYZ com are priced at $10 each and Julie thinks that they are likely to fall in price in the near future. Julie could then arrange to short sell 1,000 shares at $10, ie a total of $10,000. If she is proved correct and the shares fall in price to $9 each, then she could buy them back for $9,000, thereby netting a profit of $1,000, excluding costs.

In fact, a short sale involves borrowing the asset now, since it is not currently owned by the seller. This is usually done through a broker who is holding the asset for another client. The broker is willing to lend the asset to the short seller since the short seller promises to buy the asset back later and return the asset to the broker, which essentially returns the asset to its original owner. In addition, the short seller will normally pay a commission to the broker.

One potential complication with short sales is when the asset pays a dividend during the short sale period. The original owner of the asset is entitled to receive this dividend, but the asset is not in the owner’s brokerage account since the short seller has borrowed the asset and sold it. So the short seller must pay any dividend during the short sale period to the original owner of the asset to keep the original owner’s position whole.

Short selling an asset is like borrowing cash, a form of financing, since the short seller receives cash now and must pay at least some of it back later (and possibly more), depending upon the future price of the asset.
The lease rate of an asset

The lease rate of an asset refers to the payment made by the borrower of an asset to the lender of the asset.

So, the dividend payment from the short seller to the original owner of a share is an example of the lease rate of the asset, here the share. In fact, any payment from the borrower (short seller) to the lender (broker or original owner of the asset) is a lease rate payment of the asset.

Credit risk in short selling

With short selling, there is always a risk that the short seller may be unable to buy back the asset at a later date and return it to the original owner, especially if the asset price rises during the short sale period. This risk is called credit risk, since the short seller is a creditor of the original owner, and will be of concern to the broker who agreed to the short sale.

To mitigate this risk, the broker requires the short seller to deposit collateral to help cover any future losses. This collateral is called the margin and it is deposited into a margin account with the broker. The higher the margin, the greater the protection it offers against the credit risk that may arise should the asset price rise during the short sale period. If the margin is greater than the price at which the asset is sold by the short seller, the excess amount over the sale price is called the haircut.

The margin requirement is often expressed as a percentage of the sale price.

Example

For example, suppose that when Julie short sold the 1,000 XYZ com shares worth $10,000, she was required to deposit margin of 110% of the sale price. This would have corresponded to a margin deposit of $11,000 and a haircut of $1,000.

Scarcity in short selling

Since the short seller has deposited collateral in the form of margin with the broker, the short seller may expect the broker to pay interest on the margin during the short sale period. The margin interest rate that the broker pays the short seller is called the repo rate in the bond market and the short rebate in the stock market.

If the asset is scarce or in high demand, the broker may only be willing to pay a low interest rate. If the asset is not scarce, the broker may offer a higher interest rate. The difference between the market interest rate and the margin interest rate is essentially a cost to the short seller and additional compensation to the broker.

Example

So, for example, if the short rebate on Julie’s margin deposit is 4%, whereas the market interest rate is 5%, then the 1% difference represents an additional cost to Julie and a fee to the broker.
**Short interest**

Finally, the short interest measure provides an indication of how other investors view the future prospects of an asset.

*Short interest* can be expressed either as a number or a percentage. As a number, short interest is the absolute number of shares of a stock that has been sold short and not yet closed. Alternatively, as a percentage, short interest is the total number of shorted shares divided by the total number of shares issued by the underlying company.

Recall that a stock may be sold short if the investor thinks the stock price will go down. Consequently, if the short interest measure decreases significantly, it is an indication that investors are becoming less bearish on the stock, since the number of shorted shares has declined. If the short interest measure increases significantly, it is an indication that investors are becoming more bearish on the stock, since the number of shorted shares has increased.

**Short sale summary**

The short seller borrows a number of shares from a broker and deposits collateral in a margin account with the broker. The short seller sells the borrowed shares at the initial market price of the shares, which the short seller hopes will be higher than the future price of the shares. After a period of time, the short seller *closes* the short position by buying the shares back at the future price of the shares. If the future price of the shares is less than the initial price, the short seller has earned a profit (excluding costs), based on the fall in the share price. Otherwise, the short seller has incurred a loss. After the shares are repurchased, the short seller returns the shares to the broker and the short position has been *covered*. The money in the margin account is then returned to the short-seller.

> *Short sales are also covered in Chapter 6 of BPP’s text ‘Financial Mathematics’.*
Study Session 9 – Practice Questions

Question 9.1
You short sell 500 shares of a stock whose bid and ask prices are $50.24 and $50.48 respectively.
What is the dollar spread on 500 shares?

Question 9.2
Ninety days after the transaction described in Question 9.1 you cover the short position. At this time the bid and ask prices are $48.12 and $48.36 respectively.
What profit have you made after 90 days?

Question 9.3
Commission of 0.25% is paid on the selling and closing transactions in Questions 9.1 and 9.2.
How is your profit affected by the payment of these commissions?

Question 9.4
ABC com has one million shares in issue, 20,000 of which are owned by Investor A. Investor B borrows 5,000 shares from Investor A and short sells them to Investor C. At the same time, Investor D borrows 10,000 shares from Investor A and sells them to Investor E. A week later, Investor B buys 2,500 ABC shares and returns them to Investor A.
Determine the resulting short interest in ABC shares, assuming there has been no other short selling.

Question 9.5
Which of the following statements about the short sale of a bond are true?

I. Short selling is like borrowing cash.
II. The short seller is bullish on the bond.
III. Coupon payments to the original owner of the bond represent the lease rate of the bond.

(A) I only
(B) I and II only
(C) I and III only
(D) I, II and III
(E) The correct answer is not given by (A), (B), (C) or (D).
**Short straddle – example**

An investor expects that a certain stock will experience less price volatility over the next year than is currently priced into its options. The stock is currently priced at $100. The investor sells a 1-year call and sells a 1-year put with the same strike price of $100. The annual effective risk-free rate is 4%. The call premium is $10.35 and the put premium is $6.50. Draw the payoff and profit graphs of this position at option expiration.

**Solution**

The table of the potential future payoffs and profits at option expiration is as follows.

<table>
<thead>
<tr>
<th>Stock price at T = 1</th>
<th>Short straddle payoff</th>
<th>Short straddle profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>96</td>
<td>-4</td>
<td>13.52</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
<td>17.52</td>
</tr>
<tr>
<td>104</td>
<td>-4</td>
<td>13.52</td>
</tr>
<tr>
<td>108</td>
<td>-8</td>
<td>9.52</td>
</tr>
<tr>
<td>112</td>
<td>-12</td>
<td>5.52</td>
</tr>
</tbody>
</table>

Let’s verify the short straddle payoff when the stock price at expiration is $104. The short straddle payoff is the short call payoff plus the short put payoff:

\[-\text{Max}(0, 104 - 100) - \text{Max}(0, 100 - 104) = -4 - 0 = -4\]

Let’s verify the short straddle profit when the stock price at expiration is $104. The short straddle profit is the short call profit plus the short put profit:

\[-\text{Max}(0, 104 - 100) + 10.35(1.04) + \text{Max}(0, 100 - 104) + 6.50(1.04)] = 6.76 + 6.76 = 13.52\]

This short straddle receives $10.35 for the short call and $6.50 for the short put, so the net initial credit is $16.85.

Now we can graph the short straddle payoff and profit lines at option expiration.

**Short straddle – profit**

- The maximum profit for a short straddle is the future value of the call and put premiums, i.e. $FV_T(P_0) + FV_T(C_0)$ and this occurs when the stock price at expiry equals the strike price.
- The maximum loss for the short straddle is potentially unlimited.
• The breakeven price on the upside is the strike price plus the future value of both premiums, \( K + FV_T(P_V) + FV_T(C_V) \), whereas on the downside it is the strike price minus the future value of both premiums, \( K - FV_T(P_V) - FV_T(C_V) \).

The second nondirectional trade we consider is the strangle.

**Purchased or long strangle**

A purchased or long strangle is like a long straddled except that the strike price of the put is less than the strike price of the call. It therefore consists of long positions in both a higher strike price call and a lower strike price put.

Recall that the premium of a call is lower for a higher strike price, whilst that for a put is lower for a lower strike price. Consequently, a strangle is likely to be cheaper than a similar straddle.

**Long strangle – payoffs**

The payoffs at option expiration of a higher strike long call, a lower strike long put, and the combined position are shown below.

**Long strangle – example**

An investor expects that a certain stock will experience greater price volatility over the next year than is currently priced into its options. The stock is currently priced at $100. The investor buys a $108 strike 1-year call and buys a $100 1-year put. The annual effective risk-free rate is 4%. The call premium is $6.82 and the put premium is $6.50. Draw the payoff and profit graphs of this position at option expiration.

**Solution**

Let’s create a table of the potential future payoffs and profits at option expiration.

<table>
<thead>
<tr>
<th>Stock price at ( T = 1 )</th>
<th>Long strangle payoff</th>
<th>Long strangle profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>96</td>
<td>4</td>
<td>-9.85</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
<td>-13.85</td>
</tr>
<tr>
<td>104</td>
<td>0</td>
<td>-13.85</td>
</tr>
<tr>
<td>108</td>
<td>0</td>
<td>-13.85</td>
</tr>
<tr>
<td>112</td>
<td>4</td>
<td>-9.85</td>
</tr>
</tbody>
</table>
Let’s verify the long strangle payoff when the stock price at expiration is $104. The long strangle payoff is the higher strike long call payoff plus the lower strike long put payoff:

\[
\text{Max}(0, 104 - 108) + \text{Max}(0, 100 - 104) = 0 + 0 = 0
\]

Let’s verify the long strangle profit when the stock price at expiration is $104. The long strangle profit is the higher strike long call profit plus the lower strike long put profit:

\[
[\text{Max}(0, 104 - 108) - 6.82(1.04)] + [\text{Max}(0, 100 - 104) - 6.50(1.04)] = -7.09 - 6.76 = -13.85
\]

Notice that this long strangle requires paying $6.82 for the higher strike long call and paying $6.50 for the lower strike long put, so the net initial debit is $13.32, which is less than the net initial debit of $16.85 for the long straddle.

Now we can graph the long strangle payoff and profit lines at option expiration.

A long strangle can also be created by purchasing a lower strike call and a higher strike put. In this case, the payoffs at option expiration are shown in the graphs below.

**Long strangle – profit**

For a long strangle consisting of long positions in a higher strike price call and a lower strike price put:

- The *maximum profit* for the long strangle is potentially unlimited.
- The *maximum loss* is limited to the future value of both premiums, i.e. \( FV_T(C_0^{K_1}) + FV_T(P_0^{K_2}) \).
- The *upside breakeven price* at expiry is the higher strike plus the future value of the premiums, i.e. $K_2 + FV_T(P_0^{K_1}) + FV_T(C_0^{K_2})$, and the *downside breakeven price* is the lower strike price minus the future value of the premiums, i.e. $K_1 - FV_T(P_0^{K_1}) - FV_T(C_0^{K_2})$.

By comparing the profit in the above example with that from the straddle considered earlier, you can see that whilst a strangle is cheaper, it will produce:

- a lower profit than a similar straddle if the stock price does move considerably, either upwards or downwards
- a loss over a wider range of values if the stock price doesn’t move much.

**Written or short strangle**

The *written or short strangle* position is the opposite of the purchased strangle.

A written strangle might be used by an investor who expects less price volatility in the underlying asset over the investment horizon than is currently priced into the options. Note that the investor receives both the call and put premiums.

**Short strangle – payoffs**

The payoffs at option expiration of a higher strike short call, a lower strike short put, and the combined position are shown below.

![Payoff Graphs](image)

**Short strangle – example**

An investor expects that a certain stock will experience less price volatility over the next year than is currently priced into its options. The stock is currently priced at $100. The investor sells a $108 strike 1-year call and sells a $100 1-year put. The annual effective risk-free rate is 4%. The call premium is $6.82 and the put premium is $6.50. Draw the payoff and profit graphs of this position at option expiration.

**Solution**

The table of the potential future payoffs and profits at option expiration is as follows.
In practice, arbitrage opportunities do not exist for long since once they are recognized, market forces cause the observed price to move toward and equal the fair price, at which point arbitrage is no longer possible.

### 13.3 Stock forwards

We have already discussed forward contracts in Study Session 10. A **forward price** is a price agreed upon now for a transaction scheduled to take place at a specified time in the future. Recall that investors can enter into a forward agreement as buyers or sellers of the underlying asset. In either case, there is no initial cost to entering into a forward agreement (if we ignore the off-market forwards we discussed briefly in Study Session 10). This is in contrast to the prepaid forward contracts, which must be purchased for the prepaid forward price.

If the forward price agreed upon now for the purchase of a share of stock at time $T$ is $F_{0,T}$, then the value of the forward agreement at time $T$ for a buyer is the opposite of the value to the seller:

Payoff to the buyer of the forward = $S_T - F_{0,T}$
Payoff to the seller of the forward = $F_{0,T} - S_T$

Notice that the payoff to a forward agreement can be positive or negative, in contrast to call and put options, which can never have negative payoffs.

Just as we saw with prepaid forward contracts, there are three formulas for the forward price of an asset, depending on whether and how dividends are paid by the underlying stock. Since the forward price is not paid until time $T$, the forward price is the accumulated value of the prepaid forward price for each of the following cases:

1. **The asset does not pay dividends**
   
   The forward price is the future value of the initial stock price, accumulated at the risk-free interest rate:

   $$F_{0,T} = FV_{0,T}(F_{0,T}) = S_0 e^{rT}$$

2. **The asset pays discrete dividends**
   
   The forward price is the future value of the initial stock price less the future value of the accumulated discrete dividend payments:

   $$F_{0,T} = FV_{0,T}[S_0 - PV_{0,T}(Div)] = S_0 e^{rT} - FV_{0,T}(Div)$$

3. **The asset pays continuous dividends**
   
   The forward price is the future value of the initial stock price less the future value of the continuously paid dividend payments:

   $$F_{0,T} = FV_{0,T}[S_0 e^{-\delta T}] = S_0 e^{(r-\delta)T}$$

The forward price relationships are summarized in the following box.
**Forward price**

The forward price agreed upon now for delivery (and payment) at time $T$ depends on the dividends paid to the stockholders:

- No dividends: $F_{0,T} = S_0 e^{rT}$
- Discrete dividends: $F_{0,T} = S_0 e^{rT} - FV_{0,T}(Div)$
- Continuous dividends: $F_{0,T} = S_0 e^{(r-\delta)T}$

Remember that a fairly priced forward contract has zero initial cost, so its *premium* is zero. The forward price, which is paid at time $T$, is determined according to the above formulas.

We can also define the *forward premium*. This is the forward price divided by the underlying asset's initial price. We can also express the forward premium on an annualized basis as a percentage.

**Forward premiums**

Forward premium = \[ \frac{F_{0,T}}{S_0} \]

Annualized forward premium = \[ \frac{1}{T} \ln \left( \frac{F_{0,T}}{S_0} \right) \]

for discretely paid dividends

= $r - \delta$ for continuously paid dividends

→ *Take care that you don’t muddle up premiums and prices when reading exam questions!*

**An interpretation of the forward pricing formulas**

The difference in the forward price and the spot price of an asset reflects the cost of carry. The *cost of carry* is the net expense incurred while an investor has a position in a security. It is the cost of holding, or carrying, the security.

1. For a long position in a security:

   The cost of carry reflects the opportunity cost of buying the particular security rather than something else. For most long positions, the cost of carry is the risk-free interest rate that could have been earned less any income earned from the investment, such as dividends. So, when the investment is a share of stock, the cost of carry (per unit of investment) is the risk-free interest rate less the dividend yield: $r - \delta$. The dividends received offset the opportunity cost of the interest foregone.

2. For a short position in a security:

   The cost of carry reflects the income that the short seller of an asset owes to the owner of the asset; in the case of a stock, the dividend yield, $\delta$. In this context, the dividend yield is the *lease rate* of the stock. (Recall from Study Session 9 that the lease rate is the payment rate that the borrower (ie short seller) pays to the lender (ie original owner)). However, the short seller will receive interest on the proceeds of the short sale. The cost of carry (per unit of investment) is therefore $\delta - r$. 
3. For a forward contract:

There is no initial cost to enter into a forward agreement, so there is no opportunity cost. Also, asset ownership with a forward does not transfer until time $T$, so there is no payment required over this period even if the underlying asset pays dividends. In other words, the lease rate of a forward contract is zero. Consequently, the cost of carry is zero for a forward contract, unlike a position in the underlying asset itself. The difference between the forward price and the spot price of an asset, therefore reflects the cost of carry.

**Synthetic forwards**

We now turn our attention from forward contracts and prices to synthetic forward contracts. It is important to be able to create synthetic long forward and synthetic short forward contracts, both for taking advantage of potential arbitrage opportunities and for taking an offsetting position in an actual forward contract.

Recall from Study Session 10 that a synthetic long forward position can be created by buying the underlying asset and selling a zero-coupon bond (i.e. borrowing), as demonstrated by their payoff diagrams:

To duplicate the cash flows of a long forward, we recall that a long forward has a cash flow of 0 at time 0 and a payout of $S_T - F_{0,T}$ at time $T$. How do we replicate these cash flows using a long stock position and short bond position?

Assuming continuously paid dividends that are reinvested in the stock, the investor would need to buy $S_0 e^{-\delta T}$ shares at time 0 to result in one share of stock worth $S_T$. In addition, the investor would need to borrow $F_{0,T} = S_0 e^{-\delta T}$ at time 0, by issuing a zero-coupon bond, so that he needs to repay $F_{0,T} = S_0 e^{(r - \delta) T}$ at time $T$. This combination of buying a stock and issuing a zero-coupon bond exactly replicates the payoff of a long forward.

$$[\text{Long forward}] = [\text{stock}] - [\text{zero-coupon bond}]$$

Thus, to replicate the payoff of a long forward, the investor should buy $e^{-\delta T}$ shares of the stock and borrow $S_0 e^{-\delta T}$. 
<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0 cash flows</th>
<th>Time T cash flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy $e^{-\delta T}$ shares of stock</td>
<td>$-S_0e^{-\delta T}$</td>
<td>$S_T$</td>
</tr>
<tr>
<td>Borrow $S_0e^{-\delta T}$</td>
<td>$S_0e^{-\delta T}$</td>
<td>$-S_0e^{(r-\delta)T}$</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>$S_T - S_0e^{(r-\delta)T} = S_T - F_{0,T}$</td>
</tr>
</tbody>
</table>

Notice that we can equally create a synthetic long stock simply by rearranging the long forward equation.

$$[\text{Stock}] = [\text{long forward}] + [\text{zero-coupon bond}]$$

To replicate the payoff of a long stock, the investor should go long one forward contact and lend $S_0e^{-\delta T}$ (i.e., buy a zero-coupon bond).

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0 cash flows</th>
<th>Time T cash flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long one forward</td>
<td>0</td>
<td>$S_T - F_{0,T}$</td>
</tr>
<tr>
<td>Lend $S_0e^{-\delta T}$</td>
<td>$-S_0e^{-\delta T}$</td>
<td>$S_0e^{(r-\delta)T}$</td>
</tr>
<tr>
<td>Total</td>
<td>$-S_0e^{-\delta T}$</td>
<td>$S_T - S_0e^{(r-\delta)T} + F_{0,T} = S_T$</td>
</tr>
</tbody>
</table>

Finally, we can also create a synthetic long zero-coupon bond.

$$[\text{zero-coupon bond}] = [\text{stock}] - [\text{long forward}]$$

Here the investor should buy $e^{-\delta T}$ shares of the stock and short one forward. The implied yield on this synthetic zero-coupon bond is called the implied repo rate.

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0 cash flows</th>
<th>Time T cash flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy $e^{-\delta T}$ shares of stock</td>
<td>$-S_0e^{-\delta T}$</td>
<td>$S_T$</td>
</tr>
<tr>
<td>Short one forward</td>
<td>0</td>
<td>$F_{0,T} - S_T$</td>
</tr>
<tr>
<td>Total</td>
<td>$-S_0e^{-\delta T}$</td>
<td>$F_{0,T}$</td>
</tr>
</tbody>
</table>

Recall that the zero-coupon bond involves lending (or borrowing) $S_0e^{-\delta T}$ at time 0 and receiving (paying) back $F_{0,T} = S_0e^{(r-\delta)T}$ at time $T$.

The corresponding short positions of the above synthetic assets are created by taking the opposite sides of the above transactions.

### 13.4 Application to market-making and arbitrage

Once we become familiar with the above relationships, we can combine them in other ways as well. We are particularly interested in how a market-maker might eliminate his exposure to price risk by creating the opposite side of a position, or how an arbitrageur might identify a mispriced asset and take advantage of an arbitrage opportunity.
14.0 Swap overview

Swaps are agreements under which counterparties agree to exchange (swap) specific future cash flows. For example, in an interest rate swap, the counterparties agree to exchange future interest payments. This provides a way for a company to hedge a series of uncertain future payments.

Swaps involve a series of payments over a period of time. This makes them equivalent to a series of forward contracts. Since the counterparties are exchanging cash flows, it is often the case that the cash flows will be netted off, one against the other, so that only one cash flow takes place on each settlement date.

Historically, these arrangements were first developed as corporate-to-corporate deals with any financial intermediary simply acting as a broker. Now, the financial intermediaries act as market-makers. They execute swaps where there is no counterparty as yet, holding one side, or leg, of the deal on their own account (called warehousing one leg of the transaction) until a counterparty can be found. In this way, they face the true risk of an intermediary. The market-makers aim to build a book of transactions that offset each other overall, if offsetting does not happen on a deal-by-deal basis.

Swaps are over-the-counter (OTC) instruments, as opposed to exchange-traded instruments. This means that their terms are flexible and can be tailored to the requirements of the investor, rather than standardized. In addition, there is no daily mark-to-market procedure and no clearinghouse acting as counterparty to the deal, meaning that credit risk is greater than for an exchange-traded product.

The fact that swaps are OTC traded also means that they are less heavily regulated than exchange-traded products, a fact often perceived as a benefit by market participants.

The impact of greater credit risk and less regulation has been that the swaps market is effectively restricted to institutions and companies. These participants are more sophisticated than private investors and their credit risk is easier to monitor.

14.1 Interest rate swaps

An interest rate swap is a contract that commits two counterparties to exchange two streams of interest payments over an agreed period (known as the swap term), each calculated using a different interest rate index, but applied to a common notional principal amount. In addition, the payments are made in the same currency (unlike a currency swap).

Often the swap agreement is to pay or receive the difference between a fixed interest rate and a floating interest rate, based on the notional principal amount. They may therefore be undertaken to change exposure to fixed and floating interest rates, as well as for other reasons, which we will discuss later in the Study Session. For the time being, however, it is this type of interest rate swap that we will consider for the remainder of this section.

There are two counterparties to an interest rate swap: a fixed-rate payer who pays a fixed interest rate and receives a floating interest rate, and a floating-rate payer who pays a floating interest rate and receives a fixed interest rate.

The interest payments are paid at the end of each period, based on the interest rate at the start of the period (we’ll say more about this later on) and are based on a notional amount of principal.
Only interest payments are exchanged in the swap. The notional principal itself isn’t exchanged, as to do so would involve swapping the same monetary amount in the same currency. Interest rate swaps do not, therefore, have an effect on the balance sheet, only on the income (profit and loss) statement. Hence, they are classified as off-balance sheet instruments.

Movements in the interest cash flow streams take place at intervals during the swap’s life and are normally netted. For example, if the fixed rate exceeds the floating rate, then the fixed-rate payer pays the difference (multiplied by the notional amount). This reduces the credit risk between the counterparties.

In addition to the counterparties, there is usually a dealer, typically an investment bank, which arranges the swap and acts as the intermediary. The dealer profits from the swap by earning a spread, i.e. providing a lower fixed interest rate to the floating-rate payer than is received from the fixed-rate payer.

The diagram below illustrates a swap in which the fixed-rate payer pays 6.8% and receives 1-year LIBOR (London Interbank Offer Rate), and the floating-rate payer pays 1-year LIBOR and receives 6.7% fixed. The dealer’s spread here is 0.1%.

Interest swaps can be difficult to understand at first, so let’s look at a couple of examples.

The first example is based on the situation shown in the diagram above. The second example shows how a company with a mismatch between its assets and liabilities can use an interest rate swap to adjust its portfolio.

**Example 1**

Company A enters into an annual payment 4-year swap as a fixed-rate payer, paying 6.8% and receiving 1-year LIBOR.

Company B enters into an annual payment 4-year swap as a floating-rate payer, receiving 6.7% fixed and paying 1-year LIBOR.

The floating rate payments are determined by 1-year LIBOR on the payment date. The notional principal amount is $1,000,000.

1-year LIBOR develops as shown in the table below, which shows the interest rate at the start of each year:

<table>
<thead>
<tr>
<th>Year</th>
<th>1-year LIBOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.0%</td>
</tr>
<tr>
<td>2</td>
<td>8.0%</td>
</tr>
<tr>
<td>3</td>
<td>6.7%</td>
</tr>
<tr>
<td>4</td>
<td>6.0%</td>
</tr>
</tbody>
</table>

Identify the cash flows to/from each party during the 4-year term of the swap.
Solution

The table below shows the cash flows received and paid by both Company A and Company B. Only the net cash flow payments are actually made. Each year, the net payment is exchanged at the end of the year, based on the 1-year LIBOR rate at the start of that year.

When LIBOR is above 6.8%, the fixed-rate payer receives payments from the dealer, shown as positive numbers in Column (5) below. When LIBOR is below 6.8%, the fixed-rate payer (Company A) makes payments to the dealer, shown as negative numbers in Column (5).

<table>
<thead>
<tr>
<th>(1) Year</th>
<th>(2) 1-year LIBOR</th>
<th>(3) Receives</th>
<th>(4) Pays</th>
<th>(5) Net payment received</th>
<th>(6) Receives</th>
<th>(7) Pays</th>
<th>(8) Net payment received</th>
<th>(9) Receives</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.0%</td>
<td>70,000</td>
<td>68,000</td>
<td>2,000</td>
<td>67,000</td>
<td>70,000</td>
<td>-3,000</td>
<td>1,000</td>
</tr>
<tr>
<td>2</td>
<td>8.0%</td>
<td>80,000</td>
<td>68,000</td>
<td>12,000</td>
<td>67,000</td>
<td>80,000</td>
<td>-13,000</td>
<td>1,000</td>
</tr>
<tr>
<td>3</td>
<td>6.7%</td>
<td>67,000</td>
<td>68,000</td>
<td>-1,000</td>
<td>67,000</td>
<td>67,000</td>
<td>0</td>
<td>1,000</td>
</tr>
<tr>
<td>4</td>
<td>6.0%</td>
<td>60,000</td>
<td>68,000</td>
<td>-8,000</td>
<td>67,000</td>
<td>60,000</td>
<td>7,000</td>
<td>1,000</td>
</tr>
</tbody>
</table>

When LIBOR is above 6.7%, the floating-rate payer (Company B) makes payments to the dealer, shown as negative numbers in Column (8). When LIBOR is below 6.7%, the floating-rate payer receives payments from the dealer, shown as positive numbers Column (8).

Regardless of LIBOR's level, the dealer receives payments of $1,000 (Column (9)), which is equal to 10 basis points (i.e., 0.1%) times the notional principal of $1,000,000.

Example 2

This example illustrates how a company can use a swap to remove a mismatch between its assets and its liabilities, in this case between fixed-rate assets and floating-rate liabilities.

Company C has an obligation to pay its policyholders 1-year LIBOR plus 10 basis points for the next 4 years. The portfolio backing this liability is invested in 4-year bonds that pay a fixed rate of 7.0%.

(i) Which poses a problem for the company: interest rates moving up or interest rates moving down?

(ii) In order to reduce its risk, the company makes use of the swap described in the previous example. Does the company choose to be a fixed-rate payer or a floating-rate payer?

(iii) After entering into the swap, what spread does the company earn above the interest rate paid to its policyholders?

Solution

(i) Company C is exposed to the risk that interest rates move up. If 1-year LIBOR increases beyond 6.9%, then the company's asset portfolio will not provide enough income to cover the company's obligations to its policyholders.

(ii) Company C enters into the swap as a fixed-rate payer. It pays a fixed rate (financed by the interest received from its portfolio) and receives a floating-rate, which it uses to pay its obligation to its policyholders.
The company earns a spread of 10 basis points above the interest rate paid to policyholders. It receives 7.0% from its bond, and it pays out 6.8% to the dealer, leaving it with 20 basis points. Of this spread, 10 basis points must be added to the LIBOR payments received from the dealer to be passed on to the policyholders, leaving 10 basis points for the company.

The diagram below illustrates the cash flows coming into and flowing out of the company described in the previous example.

![Diagram]

Diagrams such as this are often useful in the exam to visualize the cash flows and calculate the net payments for a particular party.

The swap has therefore enabled the company to match up its asset and liability cash flows and so remove its exposure to the risk that interest rates increase. Regardless of how interest rates actually move, it can guarantee to always make an overall return of 10 basis points, as:

\[
+7.0 - 6.8 + \text{LIBOR} - (\text{LIBOR} + 0.10) = +0.10\%
\]

Removing a mismatch between asset and liability cash flows is one of the main uses of swaps. Here the mismatch was between two different interest rates and so an interest rate swap was used. Conversely, had the mismatch involved two different currencies, then a currency swap (swapping payments in one currency for payments in another currency) would be used. In principal, it is possible to design and use an appropriate swap to match up any different sets of cash flows.

**Example 3**

A further important use of swaps is for the purpose of hedging against the risk that market conditions move in an unfavorable way. In this example, the company uses an interest rate swap to hedge against the risk that interest rates rise. In a similar way, other types of swaps can be used to hedge against changes in other market conditions, eg price rises or currency movements.

Let’s assume that Company D borrowed $5 million on a floating-rate basis in the past, paying LIBOR+2% (assume an annual coupon). The maturity of the debt is now five years.