SOA Exam M
Life Contingencies
Flashcards

Fall 2011 exams

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HOW TO USE THESE FLASHCARDS

These flashcards are designed to help you to prepare efficiently in the run-up to the Life Contingencies segment (**MLC**) of Course M exam of the Society of Actuaries. They include conceptual ideas, key formulas and techniques for efficient problem solving. **MLC** has a number of problems that require first principles reasoning as well as a fair amount of computation. So don't look at the lists of formulas as simply being memorization work. There are often simple intuitive ideas that underlie the formulas as well as basic mathematical reasons why they are correct. Strive to understand and learn the key relations from these points of view and your knowledge will not be the superficial type that may collapse under the stress of taking the examination. The more that you understand, the easier it becomes to retain the key ideas and write them down quickly and accurately.

We have designed the flashcards so that they can be carried conveniently and read frequently in the final run-up to the exam, *eg* when commuting to work. We hope that you will personalize them by adding your own comments and notes, and checking each section when you feel confident with the material covered.

You will probably also find these summaries useful when you are at the stage of working through the past exams. If you see a particular point being examined that is not summarized here add it to these flashcards. Let us know if you find some key ideas that are missing.

Good luck with your studying.

Tabular multiple decrement models

In a tabular model you are given a joint distribution of $K = [T]$, the curtate time until decrement, and J. The joint probability function gives the probability that decrement occurs in year $k+1$ due to cause j:

$$\Pr(K = k \text{ and } J = j) = \frac{d_{x+k}^{(j)}}{l_x^{(\tau)}} = {}_{k|}q_x^{(j)}$$

From the table you can compute $q_x^{(j)}$. However, you will be unable to compute $q_x'^{(j)}$ without the help of an interpolation assumption to extend the discrete model to a continuous one. There is however one general relation between the probabilities $\left\{q_x^{(j)}\right\}$ and the absolute rates $\left\{q_x'^{(j)}\right\}$:

$$\sum_{j=1}^{r} q_x^{(j)} = q_x^{(\tau)} = 1 - p_x^{(\tau)} = 1 - \prod_{j=1}^{r} p_x'^{(j)}$$

$$= 1 - \prod_{j=1}^{r}\left(1 - q_x'^{(j)}\right) \Rightarrow$$

$$r = 2 \qquad q_x^{(1)} + q_x^{(2)} = q_x'^{(1)} + q_x'^{(2)} - q_x'^{(1)}\,q_x'^{(2)}$$

$$r = 3 \qquad q_x^{(1)} + q_x^{(2)} + q_x^{(3)} = q_x'^{(1)} + q_x'^{(2)} + q_x'^{(3)} - q_x'^{(1)}\,q_x'^{(2)} - q_x'^{(1)}\,q_x'^{(3)} - q_x'^{(2)}\,q_x'^{(3)} + q_x'^{(1)}\,q_x'^{(2)}\,q_x'^{(3)}$$

Common interpolation methods

In the following, x is an integer age and $0 \le t \le 1$.

1. The UDD assumption in each single decrement model (SUDD)

$$ {}_tq_x'^{(j)} = t \cdot q_x'^{(j)} \quad \text{or} \quad {}_tp_x'^{(j)} = 1 - t \cdot q_x'^{(j)} \Rightarrow$$

$$ {}_tp_x'^{(j)}\,\mu^{(j)}(x+t) = q_x'^{(j)} \quad \Rightarrow$$

$$ q_x^{(j)} = \int_0^1 {}_tp_x^{(\tau)} \cdot \mu^{(j)}(x+t)\,dt$$

$$ = \int_0^1 \left(\prod_{i \ne j} {}_tp_x'^{(i)}\right) \cdot \underbrace{{}_tp_x'^{(j)}\,\mu^{(j)}(x+t)}_{q_x'^{(j)}} dt = q_x'^{(j)} \int_0^1 \left(\prod_{i \ne j} {}_tp_x'^{(i)}\right) dt$$

If there are $r = 3$ modes of decrement and $\{i, j, k\} = \{1, 2, 3\}$, then the relation is:

$$ q_x^{(j)} = q_x'^{(j)}\left(1 - \frac{1}{2}\left(q_x'^{(i)} + q_x'^{(k)}\right) + \frac{1}{3}\left(q_x'^{(i)}\,q_x'^{(k)}\right)\right)$$

For the 2 decrement version set $q_x'^{(k)} = 0$.

2. The UDD assumption for each decrement in the multiple decrement table (MUDD)

$$ {}_tq_x^{(j)} = t \cdot q_x^{(j)} \text{ for all } j \quad \Rightarrow$$

$$ q_x^{(j)} = q_x^{(\tau)} \cdot \frac{\ln\left(p_x'^{(j)}\right)}{\ln\left(p_x^{(\tau)}\right)} \quad \Leftrightarrow \quad p_x'^{(j)} = \left(p_x^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}}$$

Example. Suppose that $q_x'^{(1)}=.10$, $q_x'^{(2)}=.05$. Calculate $q_x^{(1)}$.

SUDD: $q_x^{(1)}=q_x'^{(1)}\left(1-\frac{1}{2}\cdot q_x'^{(2)}\right)=.10\left(1-.5(.05)\right)=.09750$

MUDD: $q_x^{(1)}=q_x^{(\tau)}\cdot\dfrac{\ln\left(p_x'^{(1)}\right)}{\ln\left(p_x^{(\tau)}\right)}=\left(1-(.90\times.95)\right)\cdot\dfrac{\ln(.90)}{\ln(.90\times.95)}$
$=.09752$

Discrete decrement patterns

The above theory assumes that each decrement occurs continuously (T_j is a continuous random variable). The formula

$$q_x^{(j)}=\int_0^1 \underbrace{\left(\prod_{i\neq j} {}_tp_x'^{(i)}\right)}_{\substack{\text{probability of surviving other}\\ \text{decrements for } t \text{ years}}} \cdot \underbrace{{}_tp_x'^{(j)}\,\mu^{(j)}(x+t)\,dt}_{\substack{f_{T_j}(t)dt \text{ – an element}\\ \text{of } T_j \text{ probability}}}$$

can be reinterpreted to provide a method for dealing with decrement *j* when it follows a *discrete pattern*:

> If decrement j can occur at times $0\leq t_1<\cdots<t_n\leq 1$ with respective probabilities p_1 , p_2 , ... , p_n , then

$$q_x^{(j)}=\sum_{i=1}^{n}\left\{\underbrace{\prod_{k\neq j} {}_{t_i}p_x'^{(k)}}_{\substack{\text{probability of surviving}\\ \text{other decrements for } t_i \text{ years}}} \cdot \underbrace{p_i}_{\substack{\text{probability of decrement } j\\ \text{occurring at time } t_i}}\right\}$$

Example. Calculate $q_x^{(2)}$ given $q_x'^{(1)}=.10$, $q_x'^{(2)}=.08$, decrement 1 is SUDD, and decrement 2 occurs half at mid-year and half at year-end. It follows that:

$$t_1=.50\ ,\ t_2=1.00\ ,\ p_1=.08/2=.04\ ,\ p_2=.08/2=.04$$
$${}_tp_x'^{(1)}=1-t\cdot q_x'^{(1)}=1-.10t\ \Rightarrow$$
$${}_{t_1}p_x'^{(1)}={}_{.50}p_x'^{(1)}=1-.10(.50)=.95$$
$${}_{t_2}p_x'^{(1)}={}_{1.00}p_x'^{(1)}=1-.10(1.00)=.90$$
$$q_x^{(2)}={}_{t_1}p_x'^{(1)}\cdot\Pr(T_2=t_1)+{}_{t_2}p_x'^{(1)}\cdot\Pr(T_2=t_2)$$
$$=(.95)(.04)+(.90)(.04)=.074$$

Benefits depending on the time and mode of decrement

Suppose that an amount $b_{T,J}$ is paid at the time of decrement *T* if *J* is the mode of decrement. The random present value of the benefit is $Z=b_{T,J}\cdot v^T$. It is a function of *T* and *J*, so moments of *Z* can be computed as follows:

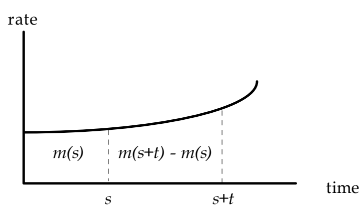
$$E\left[Z^k\right]=\sum_{j=1}^{r}\int_0^\infty\left(b_{t,j}\,v^t\right)^k f_{T,J}(t,j)\,dt$$

Example of thinning

- An insurer wishes to institute an ordinary deductible of d per loss. Let C_1 denote the category of losses that exceed the deductible, and let C_2 denote the category of losses that are less than or equal to the deductible: $p_1 = \Pr(X > d)$. Then $N_1(t)$ is a Poisson process with rate $\lambda_1 = \lambda p_1 = \lambda \Pr(X > d)$. This is the frequency of *payment* events for the insurer over a period of t years.
- Losses could be categorized by *types*. Suppose we have losses against car insurance policies and we want to eliminate broken windshield losses from coverage when 3% of claims are for broken windshields. To know the frequency of losses after this elimination define category C_1 for windshield losses, and category C_2 for all other types of losses. The Poisson process $N_2(t)$ would then have a rate equal to $\lambda_2 = \lambda p_2 = 0.97\lambda$.

Non-homogeneous Poisson process

Denote the *process rate* at time t for a non-homogeneous Poisson process by $\lambda(t)$. The *mean value function* for this process is defined by $m(t) = \int_0^t \lambda(s)\,ds$.



Properties of the non-homogenous Poisson process

1. The process increment $N(s+t) - N(s)$ is Poisson distributed with parameter (mean) equal to $m(s+t) - m(s)$.

2. Process increments corresponding to disjoint intervals are independent.

3. For a non-homogeneous Poisson process, increments corresponding to intervals of the same length do not necessarily have identical means.

4. The survival and probability density functions for the first inter-arrival time are:

$$s_{T_1}(t) = \Pr(N(t) = 0) = e^{-m(t)} \quad \text{and} \quad f_{T_1}(t) = \lambda(t) e^{-m(t)}$$

5. For $n \geq 2$, the conditional survival function and probability density function for the n-th inter-arrival time, given the $(n-1)$-th event time is $S_{n-1} = s_{n-1}$, are:

$$\begin{aligned} s_{T_n}(t \mid S_{n-1} = s_{n-1}) &= \Pr\left(N(s_{n-1} + t) - N(s_{n-1}) = 0\right) \\ &= \exp\left(-\left(m(s_{n-1} + t) - m(s_{n-1})\right)\right) \end{aligned}$$

$$f_{T_n}(t \mid S_{n-1} = s_{n-1}) = \underbrace{\lambda(s_{n-1} + t)}_{\text{force}} \underbrace{\exp\left(-\left(m(s_{n-1} + t) - m(s_{n-1})\right)\right)}_{\text{survival function}}$$

where $m(t) = \int_0^t \lambda(s)\,ds$.