Solution to *Derivatives Markets*: SOA Exam MFE and CAS Exam 3 FE

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Contents

Preface

This is Guo's solution to Derivatives Markets (2nd edition ISBN 0-321-28030-X) for SOA MFE or CAS Exam 3 FE. Unlike the official solution manual published by Addison-Wesley, this solution manual provides solutions to both the even-numbered and odd-numbered problems for the chapters that are on the SOA Exam MFE and CAS Exam 3 FE syllabus. Problems that are out of the scope of the SOA Exam MFE and CAS Exam 3 FE syllabus are excluded.

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Introduction

Recommendations on using this solution manual:

1. Obviously, you'll need to buy Derivatives Markets (2nd edition) to see the problems.
2. Make sure you download the textbook errata from `http://www.kellogg.northwestern.edu/faculty/mcdonald/htm/typos2e_01.html`

Chapter 9

Parity and other option relationships

Problem 9.1.

$S_0 = 32 \qquad T = 6/12 = 0.5 \qquad K = 35$
$C = 2.27 \qquad r = 0.04 \qquad \delta = 0.06$

$C + PV(K) = P + S_0 e^{-\delta T}$
$2.27 + 35e^{-0.04(0.5)} = P + 32e^{-0.06(0.5)} \qquad P = 5.5227$

Problem 9.2.

$S_0 = 32 \qquad T = 6/12 = 0.5 \qquad K = 30$
$C = 4.29 \qquad P = 2.64 \qquad r = 0.04$
$C + PV(K) = P + S_0 - PV(Div)$
$4.29 + 30e^{-0.04(0.5)} = 2.64 + 32 - PV(Div)$
$PV(Div) = 0.944$

Problem 9.3.

$S_0 = 800 \qquad r = 0.05 \qquad \delta = 0$
$T = 1 \qquad K = 815 \qquad C = 75 \qquad P = 45$
a. Buy stock+ sell call+buy put=buy $PV(K)$
$C + PV(K) = P + S_0$
$\rightarrow PV(K = 815) = \underbrace{S_0}_{\text{buy stock}} + \underbrace{-C}_{\text{sell call}} + \underbrace{P}_{\text{buy put}} = 800 + (-75) + 45 = 770$

So the position is equivalent to depositing 770 in a savings account (or buying a bond with present value equal to 770) and receiving 815 one year later. $770e^R = 815 \qquad R = 0.0568$

So we earn 5.68%.

b. Buying a stock, selling a call, and buying a put is the same as depositing $PV(K)$ in the savings account. As a result, we should just earn the risk free interest rate $r = 0.05$. However, we actually earn $R = 0.0568 > r$. To arbitrage, we "borrow low and earn high." We borrow 770 from a bank at 0.05%. We use the borrowed 770 to finance buying a stock, selling a call, and buying a put. Notice that the net cost of buying a stock, selling a call, and buying a put is 770.

One year later, we receive $770e^R = 815$. We pay the bank $770e^{0.05} = 809.48$. Our profit is $815 - 809.48 = 5.52$ per transaction.

If we do n such transactions, we'll earn $5.52n$ profit.

Alternative answer: we can burrow at 5% (continuously compounding) and lend at 5.68% (continuously compounding), earning a risk free 0.68%. So if we borrow \$1 at time zero, our risk free profit at time one is $e^{0.0568} - e^{0.05} = 0.007173$; if we borrow \$770 at time zero, our risk free profit at time one is $0.007173 \times 770 = 5.52$. If we borrow n dollars at time zero, we'll earn $0.007173n$ dollars at time one.

c. To avoid arbitrage, we need to have:

$$PV(K = 815) = \underbrace{S_0}_{\text{buy stock}} + \underbrace{-C}_{\text{sell call}} + \underbrace{P}_{\text{buy put}} = 815e^{-0.05} = 775.25$$

$$\rightarrow C - P = S_0 - PV(K) = 800 - 775.25 = 24.75$$

d. $C - P = S_0 - PV(K) = 800 - Ke^{-rT} = 800 - Ke^{-0.05}$

If $K = 780$ $\quad C - P = 800 - 780e^{-0.05} = 58.041$

If $K = 800$ $\quad C - P = 800 - 800e^{-0.05} = 39.016$

If $K = 820$ $\quad C - P = 800 - 820e^{-0.05} = 19.992$

If $K = 840$ $\quad C - P = 800 - 840e^{-0.05} = 0.967$

Problem 9.4.

To solve this type of problems, just use the standard put-call parity.

To avoid calculation errors, clearly identify the underlying asset.

The underlying asset is €1. We want to find the dollar cost of a put option on this underlying.

The typical put-call parity:

$C + PV(K) = P + S_0 e^{-\delta T}$

C, K, P, and S_0 should all be expressed in dollars. S_0 is the current (dollar price) of the underlying. So $S_0 = \$0.95$.

$C = \$0.0571 \qquad K = \$0.93$

δ is the internal growth rate of the underlying asset (i.e. €1). Hence $\delta = 0.04$

Since K is expressed in dollars, $PV(K)$ needs to be calculated using the dollar risk free interest $r = 0.06$.

$0.0571 + 0.93e^{-0.06(1)} = P + 0.95e^{-0.04(1)} \qquad P = \$0.0202$

Problem 9.5.

As I explained in my study guide, don't bother memorizing the following complex formula:

$C_{\$}(x_0, K, T) = x_0 K P_f\left(\frac{1}{x_0}, \frac{1}{K}, T\right)$

Just use my approach to solve this type of problems.

Convert information to symbols:

The exchange rate is 95 yen per euro. $Y95 =€1$ or $Y1 =€\frac{1}{95}$
Yen-denominated put on 1 euro with strike price Y100 has a premium Y8.763
$\rightarrow (€1 \rightarrow Y100)_0 =$Y8.763
What's the strike price of a euro-denominated call on 1 yen? $€K \rightarrow 1Y$
Calculate the price of a euro-denominated call on 1 yen with strike price $€K$
$(€K \rightarrow 1Y)_0 = €?$

$€1 \rightarrow Y100 \qquad \rightarrow \qquad €\frac{1}{100} \rightarrow Y1$

The strike price of the corresponding euro-denominated yen call is $K =€\frac{1}{100} =€0.01$

$\left(€\frac{1}{100} \rightarrow Y1\right)_0 = \frac{1}{100} \times (€1 \rightarrow Y100)_0 = \frac{1}{100}(Y8.763)$

Since $Y1 =€\frac{1}{95}$, we have:

$\frac{1}{100}(Y8.763) = \frac{1}{100}(8.763)\left(€\frac{1}{95}\right) =€9.2242 \times 10^{-4}$

$\rightarrow \left(€\frac{1}{100} \rightarrow Y1\right)_0 =€9.2242 \times 10^{-4}$

So the price of a euro-denominated call on 1 yen with strike price $K =€\frac{1}{100}$ is $€9.2242 \times 10^{-4}$

Problem 9.6.

The underlying asset is €1. The standard put-call parity is:
$C + PV(K) = P + S_0 e^{-\delta T}$
C, K, P, and S_0 should all be expressed in dollars. S_0 is the current (dollar price) of the underlying.
δ is the internal growth rate of the underlying asset (i.e. €1).
We'll solve Part b first.

b. $0.0404 + 0.9e^{-0.05(0.5)} = 0.0141 + S_0 e^{-0.035(0.5)}$ $\quad S_0 = \$0.920\,04$
So the current price of the underlying (i.e. €1) is $S_0 = \$0.920\,04$. In other words, the currency exchange rate is \$0.920 04 =€1

a. According to the textbook Equation 5.7, the forward price is:
$F_{0,T} = S_0 e^{-\delta T} e^{rT} = 0.920\,04 e^{-0.035(0.5)} e^{0.05(0.5)} = \$0.926\,97$

Problem 9.7.

The underlying asset is one yen.
a. $C + Ke^{-rT} = P + S_0 e^{-\delta T}$
$0.0006 + 0.009e^{-0.05(1)} = P + 0.009e^{-0.01(1)}$
$0.0006 + 0.008561 = P + 0.008\,91 \qquad P = \$0.00025$

b. There are two puts out there. One is the synthetically created put using the formula:

$P = C + Ke^{-rT} - S_0 e^{-\delta T}$
The other is the put in the market selling for the price for \$0.0004.

To arbitrage, build a put a low cost and sell it at a high price. At $t = 0$, we:

- Sell the expensive put for \$0.0004
- Build a cheap put for \$0.00025. To build a put, we buy a call, deposit Ke^{-rT} in a savings account, and sell $e^{-\delta T}$ unit of Yen.

		$T = 1$	$T = 1$
	$t = 0$	$S_T < 0.009$	$S_T \geq 0.009$
Sell expensive put	0.0004	$S_T - 0.009$	0
Buy call	-0.0006	0	$S_T - 0.009$
Deposit Ke^{-rT} in savings	$-0.009e^{-0.05(1)}$	0.009	0.009
Short sell $e^{-\delta T}$ unit of Yen	$0.009e^{-0.01(1)}$	S_T	S_T
Total	\$0.00015	0	0

$0.0004 - 0.0006 - 0.009e^{-0.05(1)} + 0.009e^{-0.01(1)} = \$0.00015$

At $t = 0$, we receive \$0.00015 yet we don't incur any liabilities at $T = 1$ (so we receive \$0.00015 free money at $t = 0$).

c. At-the-money means $K = S_0$ (i.e. the strike price is equal to the current exchange rate).

Dollar-denominated at-the-money yen call sells for \$0.0006. To translate this into symbols, notice that under the call option, the call holder can give \$0.009 and get $Y1$.

"Give \$0.009 and get $Y1$" is represented by (\$0.009 $\rightarrow Y1$). This option's premium at time zero is \$0.0006. Hence we have:

$(\$0.009 \rightarrow Y1)_0 = \$0.0006$

We are asked to find the yen denominated at the money call for \$1. Here the call holder can give c yen and get \$1. "Give c yen and get \$1" is represented by $(Yc \rightarrow \$1)$. This option's premium at time zero is $(Yc \rightarrow \$1)_0$.

First, we need to calculate c, the strike price of the yen denominated dollar call. Since at time zero $\$0.009 = Y1$, we have $\$1 = Y\dfrac{1}{0.009}$. So the at-the-money yen denominated call on \$1 is $c = \dfrac{1}{0.009}$. Our task is to find this option's premium: $\left(Y\dfrac{1}{0.009} \rightarrow \$1\right)_0 =?$

We'll find the premium for $Y1 \rightarrow \$0.009$, the option of "give 1 yen and get \$0.009." Once we find this premium, we'll scale it and find the premium of "give $\dfrac{1}{0.009}$ yen and get \$1."

We'll use the general put-call parity:

$(A_T \rightarrow B_T)_0 + PV(A_T) = (B_T \rightarrow A_T)_0 + PV(B_T)$

$(\$0.009 \rightarrow Y1)_0 + PV(\$0.009) = (Y1 \rightarrow \$0.009)_0 + PV(Y1)$

$PV(\$0.009) = \$0.009e^{-0.05(1)}$

Since we are discounting \$0.009 at $T = 1$ to time zero, we use the dollar interest rate 5%.

$PV(Y1) = \$0.009e^{-0.01(1)}$

If we discount Y1 from $T = 1$ to time zero, we get $e^{-0.01(1)}$ yen, which is equal to $\$0.009e^{-0.01(1)}$.

So we have:

$\$0.0006 + \$0.009e^{-0.05} = (Y1 \rightarrow \$0.009)_0 + \$0.009e^{-0.01(1)}$

$(Y1 \rightarrow \$0.009)_0 = \$2.50616 \times 10^{-4}$

$\left(\dfrac{1}{0.009}Y1 \rightarrow \$1\right)_0 = \dfrac{1}{0.009}(Y1 \rightarrow \$0.009)_0 = \$\dfrac{2.50616 \times 10^{-4}}{0.009} = \$2.78462 \times 10^{-2} = Y\dfrac{2.78462 \times 10^{-2}}{0.009} = Y3.094$

So the yen denominated at the money call for \$1 is worth $\$2.78462 \times 10^{-2}$ or $Y3.094$.

We are also asked to identify the relationship between the yen denominated at the money call for \$1 and the dollar-denominated yen put. The relationship is that we use the premium of the latter option to calculate the premium of the former option.

Next, we calculate the premium for the yen denominated at-the-money put for \$1:

$$\left(\$ \to Y\frac{1}{0.009}\right)_0 = \frac{1}{0.009}\left(\$0.009 \to Y1\right)_0$$
$$= \frac{1}{0.009} \times \$0.0006 = \$\ 0.0666 7$$
$$= Y\ 0.0666 7 \times \frac{1}{0.009} = Y\ 7.407 8$$

So the yen denominated at-the-money put for \$1 is worth \$ 0.0666 7 or Y 7.407 8.

I recommend that you use my solution approach, which is less prone to errors than using complex notations and formulas in the textbook.

Problem 9.8.

The textbook Equations 9.13 and 9.14 are violated.

This is how to arbitrage on the calls. We have two otherwise identical calls, one with \$50 strike price and the other \$55. The \$50 strike call is more valuable than the \$55 strike call, but the former is selling less than the latter. To arbitrage, buy low and sell high.

We use T to represent the common exercise date. This definition works whether the two options are American or European. If the two options are American, we'll find arbitrage opportunities if two American options are exercised simultaneously. If the two options are European, T is the common expiration date.

The payoff is:

		T	T	T
Transaction	$t=0$	$S_T < 50$	$50 \leq S_T < 55$	$S_T \geq 55$
Buy 50 strike call	-9	0	$S_T - 50$	$S_T - 50$
Sell 55 strike call	10	0	0	$-(S_T - 55)$
Total	1	0	$S_T - 50 \geq 0$	5

At $t=0$, we receive \$1 free money.

At T, we get non negative cash flows (so we may get some free money, but we certainly don't owe anybody anything at T). This is clearly an arbitrage.

This is how to arbitrage on the two puts. We have two otherwise identical puts, one with \$50 strike price and the other \$55. The \$55 strike put is more valuable than the \$50 strike put, but the former is selling less than the latter. To arbitrage, buy low and sell high.

The payoff is:

		T	T	T
Transaction	$t = 0$	$S_T < 50$	$50 \leq S_T < 55$	$S_T \geq 55$
Buy 55 strike put	-6	$55 - S_T$	$55 - S_T$	0
Sell 50 strike put	7	$-(50 - S_T)$	0	0
Total	1	5	$55 - S_T > 0$	0

At $t = 0$, we receive \$1 free money.

At T, we get non negative cash flows (so we may get some free money, but we certainly don't owe anybody anything at T). This is clearly an arbitrage.

Problem 9.9.

The textbook Equation 9.15 and 9.16 are violated.

We use T to represent the common exercise date. This definition works whether the two options are American or European. If the two options are American, we'll find arbitrage opportunities if two American options are exercised simultaneously at T. If the two options are European, T is the common expiration date.

This is how to arbitrage on the calls. We have two otherwise identical calls, one with \$50 strike price and the other \$55. The premium difference between these two options should not exceed the strike difference $15 - 10 = 5$. In other words, the 50-strike call should sell no more than $10 + 5$. However, the 50-strike call is currently selling for 16 in the market. To arbitrage, buy low (the 55-strike call) and sell high (the 50-strike call).

The \$50 strike call is more valuable than the \$55 strike call, but the former is selling less than the latter.

The payoff is:

		T	T	T
Transaction	$t = 0$	$S_T < 50$	$50 \leq S_T < 55$	$S_T \geq 55$
Buy 55 strike call	-10	0	0	$S_T - 55$
Sell 50 strike call	16	0	$-(S_T - 50)$	$-(S_T - 50)$
Total	6	0	$-(S_T - 50) \geq -5$	-5

So we receive \$6 at $t = 0$. Then at T, our maximum liability is \$5. So make at least \$1 free money.

This is how to arbitrage on the puts. We have two otherwise identical puts, one with \$50 strike price and the other \$55. The premium difference between these two options should not exceed the strike difference $15 - 10 = 5$. In other words, the 55-strike put should sell no more than $7 + 5 = 12$. However, the 55-strike put is currently selling for 14 in the market. To arbitrage, buy low (the 50-strike put) and sell high (the 55-strike put).

The payoff is:

		T	T	T
Transaction	$t = 0$	$S_T < 50$	$50 \leq S_T < 55$	$S_T \geq 55$
Buy 50 strike put	-7	$50 - S_T$	0	0
Sell 55 strike put	14	$-(55 - S_T)$	$-(55 - S_T)$	0
Total	7	-5	$-(55 - S_T) < -5$	0

So we receive \$7 at $t = 0$. Then at T, our maximum liability is \$5. So make at least \$2 free money.

Problem 9.10.

Suppose there are 3 options otherwise identical but with different strike price $K_1 < K_2 < K_3$ where $K_2 = \lambda K_1 + (1 - \lambda) K_3$ and $0 < \lambda < 1$.

Then the price of the middle strike price K_2 must not exceed the price of a diversified portfolio consisting of λ units of K_1-strike option and $(1 - \lambda)$ units of K_3-strike option:

$$C[\lambda K_1 + (1 - \lambda) K_3] \leq \lambda C(K_1) + (1 - \lambda) C(K_3)$$
$$P[\lambda K_1 + (1 - \lambda) K_3] \leq \lambda P(K_1) + (1 - \lambda) P(K_3)$$

The above conditions are called the convexity of the option price with respect to the strike price. They are equivalent to the textbook Equation 9.17 and 9.18.

If the above conditions are violated, arbitrage opportunities exist.

We are given the following 3 calls:

Strike	$K_1 = 50$	$K_2 = 55$	$K_3 = 60$
Call premium	18	14	9.50

$\lambda 50 + (1 - \lambda) 60 = 55$
$\rightarrow \lambda = 0.5 \qquad 0.5(50) + 0.5(60) = 55$

Let's check:
$C[0.5(50) + 0.5(60)] = C(55) = 14$

$0.5C(50) + 0.5C(60) = 0.5(18) + 0.5(9.50) = 13.75$
$C[0.5(50) + 0.5(60)] > 0.5C(50) + 0.5C(60)$
So arbitrage opportunities exist. To arbitrage, we buy low and sell high.

The cheap asset is the diversified portfolio consisting of λ units of K_1-strike option and $(1 - \lambda)$ units of K_3-strike option. In this problem, the diversified portfolio consists of half a 50-strike call and half a 60-strike call.

The expensive asset is the 55-strike call.

Since we can't buy half a call option, we'll buy 2 units of the portfolio (i.e. buy one 50-strike call and one 60-strike call). Simultaneously,we sell two 55-strike call options.

We use T to represent the common exercise date. This definition works whether the options are American or European. If the options are American, we'll find arbitrage opportunities if the American options are exercised simultaneously. If the options are European, T is the common expiration date.

The payoff is:

		T	T	T	T
Transaction	$t=0$	$S_T<50$	$50\leq S_T<55$	$55\leq S_T<60$	$S_T\geq 60$
buy two portfolios					
buy a 50-strike call	-18	0	S_T-50	S_T-50	S_T-50
buy a 60-strike call	-9.5	0	0	0	S_T-60
Portfolio total	-27.5	0	S_T-50	S_T-50	$2S_T-110$
Sell two 55-strike calls	$2(14)=28$	0	0	$-2(S_T-55)$	$-2(S_T-55)$
Total	0.5	0	$S_T-50\geq 0$	$60-S_T>0$	0

$-27.5+28=0.5$
$S_T-50-2(S_T-55)=60-S_T$
$2S_T-110-2(S_T-55)=0$
So we get \$0.5 at $t=0$, yet we have non negative cash flows at the expiration date T. This is arbitrage.

The above strategy of buying λ units of K_1-strike call, buying $(1-\lambda)$ units of K_3-strike call, and selling one unit of K_2-strike call is called the butterfly spread.

We are given the following 3 puts:

Strike	$K_1=50$	$K_2=55$	$K_3=60$
Put premium	7	10.75	14.45

$\lambda 50+(1-\lambda)60=55$
$\rightarrow\lambda=0.5 \qquad 0.5(50)+0.5(60)=55$

Let's check:
$P[0.5(50)+0.5(60)]=P(55)=10.75$

$0.5P(50)+0.5P(60)=0.5(7)+0.5(14.45)=10.725$
$P[0.5(50)+0.5(60)]>.5P(50)+0.5P(60)$
So arbitrage opportunities exist. To arbitrage, we buy low and sell high.

The cheap asset is the diversified portfolio consisting of λ units of K_1-strike put and $(1-\lambda)$ units of K_3-strike put. In this problem, the diversified portfolio consists of half a 50-strike put and half a 60-strike put.

The expensive asset is the 55-strike put.

Since we can't buy half a option, we'll buy 2 units of the portfolio (i.e. buy one 50-strike put and one 60-strike put). Simultaneously,we sell two 55-strike put options.

The payoff is:

		T	T	T	T
Transaction	$t=0$	$S_T<50$	$50\leq S_T<55$	$55\leq S_T<60$	$S_T\geq 60$
buy two portfolios					
buy a 50-strike put	-7	$50-S_T$	0	0	0
buy a 60-strike put	-14.45	$60-S_T$	$60-S_T$	$60-S_T$	0
Portfolio total	-21.45	$110-2S_T$	$60-S_T$	$60-S_T$	0
Sell two 55-strike puts	$2(10.75)$	$-2(55-S_T)$	$-2(55-S_T)$	0	0
Total	0.05	0	$S_T-50\geq 0$	$60-S_T>0$	0

$-21.45+2(10.75)=0.05$
$50-S_T+60-S_T=110-2S_T$

$-21.45+2(10.75)=0.05$
$110-2S_T-2(55-S_T)=0$
$60-S_T-2(55-S_T)=S_T-50$

So we get \$0.05 at $t=0$, yet we have non negative cash flows at the expiration date T. This is arbitrage.

The above strategy of buying λ units of K_1-strike put, buying $(1-\lambda)$ units of K_3-strike put, and selling one unit of K_2-strike put is also called the butterfly spread.

Problem 9.11.

This is similar to Problem 9.10.

We are given the following 3 calls:

Strike	$K_1=80$	$K_2=100$	$K_3=105$
Call premium	22	9	5

$80\lambda+105(1-\lambda)=100$
$\rightarrow\lambda=0.2\qquad 0.2(80)+0.8(105)=100$
$C[0.2(80)+0.8(105)]=C(100)=9$

$0.2C(80)+0.8C(105)=0.2(22)+0.8(5)=8.4$
$C[0.2(80)+0.8(105)]>0.2C(80)+0.8C(105)$

So arbitrage opportunities exist. To arbitrage, we buy low and sell high.

The cheap asset is the diversified portfolio consisting of λ units of K_1-strike option and $(1-\lambda)$ units of K_3-strike option. In this problem, the diversified portfolio consists of 0.2 unit of 80-strike call and 0.8 unit of 105-strike call.

The expensive asset is the 100-strike call.

Since we can't buy a fraction of a call option, we'll buy 10 units of the portfolio (i.e. buy two 80-strike calls and eight 105-strike calls). Simultaneously,we sell ten 100-strike call options.

We use T to represent the common exercise date. This definition works whether the options are American or European. If the options are American, we'll find arbitrage opportunities if the American options are exercised simultaneously. If the options are European, T is the common expiration date.

The payoff is:

		T	T
Transaction	$t=0$	$S_T<80$	$80\leq S_T<100$
buy ten portfolios			
buy two 80-strike calls	$-2\,(22)$	0	$2\,(S_T-80)$
buy eight 105-strike calls	$-8\,(5)$	0	0
Portfolio total	-84	0	$2\,(S_T-80)$
Sell ten 100-strike calls	$10\,(9)$	0	0
Total	6	0	$2\,(S_T-80)\geq 0$

		T	T
Transaction	$t=0$	$100\leq S_T<105$	$S_T\geq 105$
buy ten portfolios			
buy two 80-strike calls	$-2\,(22)$	$2\,(S_T-80)$	$2\,(S_T-80)$
buy eight 105-strike calls	$-8\,(5)$	0	$8\,(S_T-105)$
Portfolio total	-84	$2\,(S_T-80)$	$10S_T-1000$
Sell ten 100-strike calls	$10\,(9)$	$-10\,(S_T-100)$	$-10\,(S_T-100)$
Total	6	$8\,(105-S_T)>0$	0

$-2\,(22)-8\,(5)=-44-40=-84$

$-84+10\,(9)=-84+90=6$

$2\,(S_T-80)+8\,(S_T-105)=10S_T-1000$

$2\,(S_T-80)-10\,(S_T-100)=840-8S_T=8\,(105-S_T)$

$10S_T-1000-10\,(S_T-100)=0$

So we receive \$6 at $t=0$, yet we don't incur any negative cash flows at expiration T. So we make at least \$6 free money.

We are given the following 3 put:

Strike	$K_1=80$	$K_2=100$	$K_3=105$
Put premium	4	21	24.8

$80\lambda+105\,(1-\lambda)=100$

$\rightarrow\lambda=0.2 \qquad 0.2\,(80)+0.8\,(105)=100$

$P\,[0.2\,(80)+0.8\,(105)]=P\,(100)=21$

$0.2P\,(80)+0.8P\,(105)=0.2\,(4)+0.8\,(24.8)=20.64$

$P[0.2(80)+0.8(105)] > 0.2P(80)+0.8P(105)$

So arbitrage opportunities exist. To arbitrage, we buy low and sell high.

The cheap asset is the diversified portfolio consisting of λ units of K_1-strike option and $(1-\lambda)$ units of K_3-strike option. In this problem, the diversified portfolio consists of 0.2 unit of 80-strike put and 0.8 unit of 105-strike put.

The expensive asset is the 100-strike put.

Since we can't buy half a fraction of an option, we'll buy 10 units of the portfolio (i.e. buy two 80-strike puts and eight 105-strike puts). Simultaneously,we sell ten 100-strike put options.

The payoff is:

		T	T
Transaction	$t=0$	$S_T<80$	$80\leq S_T<100$
buy ten portfolios			
buy two 80-strike puts	$-2(4)$	$2(80-S_T)$	0
buy eight 105-strike puts	$-8(24.8)$	$8(105-S_T)$	$8(105-S_T)$
Portfolio total	-84	$1000-10S_T$	$8(105-S_T)$
Sell ten 100-strike puts	$10(21)$	$-10(100-S_T)$	$-10(100-S_T)$
Total	3.6	0	$2(S_T-80)\geq 0$

		T	T
Transaction	$t=0$	$100\leq S_T<105$	$S_T\geq 105$
buy ten portfolios			
buy two 80-strike puts	$-2(4)$	0	0
buy eight 105-strike puts	$-8(24.8)$	$8(105-S_T)$	0
Portfolio total	-84	$8(105-S_T)$	0
Sell ten 100-strike puts	$10(21)$	0	0
Total	3.6	$8(105-S_T)>0$	0

$-2(4)-8(24.8)=-206.4$
$2(80-S_T)+8(105-S_T)=1000-10S_T$
$-206.4+10(21)=3.6$
$1000-10S_T-10(100-S_T)=0$
$8(105-S_T)-10(100-S_T)=2(S_T-80)$

We receive \$3.6 at $t=0$, but we don't incur any negative cash flows at T. So we make at least \$3.6 free money.

Problem 9.12.

For two European options differing only in strike price, the following conditions must be met to avoid arbitrage (see my study guide for explanation):

$0\leq C_{Eur}(K_1,T)-C_{Eur}(K_2,T)\leq PV(K_2-K_1)$ if $K_1<K_2$

$0 \leq P_{Eur}(K_2, T) - P_{Eur}(K_1, T) \leq PV(K_2 - K_1)$ if $K_1 < K_2$

a.

Strike	$K_1 = 90$	$K_2 = 95$
Call premium	10	4

$C(K_1) - C(K_2) = 10 - 4 = 6$
$K_2 - K_1 = 95 - 90 = 5$
$C(K_1) - C(K_2) > K_2 - K_1 \geq PV(K_2 - K_1)$
Arbitrage opportunities exist.

To arbitrage, we buy low and sell high. The cheap call is the 95-strike call; the expensive call is the 90-strike call.

We use T to represent the common exercise date. This definition works whether the two options are American or European. If the two options are American, we'll find arbitrage opportunities if two American options are exercised simultaneously. If the two options are European, T is the common expiration date.

The payoff is:

		T	T	T
Transaction	$t = 0$	$S_T < 90$	$90 \leq S_T < 95$	$S_T \geq 95$
Buy 95 strike call	-4	0	0	$S_T - 95$
Sell 90 strike call	10	0	$-(S_T - 90)$	$-(S_T - 90)$
Total	6	0	$-(S_T - 90) \geq -5$	-5

We receive \$6 at $t = 0$, yet our max liability at T is -5. So we'll make at least \$1 free money.

b.

$T = 2$ $r = 0.1$

Strike	$K_1 = 90$	$K_2 = 95$
Call premium	10	5.25

$C(K_1) - C(K_2) = 10 - 5.25 = 4.75$
$K_2 - K_1 = 95 - 90 = 5$
$PV(K_2 - K_1) = 5e^{-0.1(2)} = 4.094$
$C(K_1) - C(K_2) > PV(K_2 - K_1)$
Arbitrage opportunities exist.

Once again, we buy low and sell high. The cheap call is the 95-strike call; the expensive call is the 90-strike call.

The payoff is:

		T	T	T
Transaction	$t = 0$	$S_T < 90$	$90 \leq S_T < 95$	$S_T \geq 95$
Buy 95 strike call	-5.25	0	0	$S_T - 95$
Sell 90 strike call	10	0	$-(S_T - 90)$	$-(S_T - 90)$
Deposit 4.75 in savings	-4.75	$4.75e^{0.1(2)}$	$4.75e^{0.1(2)}$	$4.75e^{0.1(2)}$
Total	0	5.80	$95.80 - S_T > 0$	0.80

$4.75e^{0.1(2)} = 5.80$
$-(S_T - 90) + 4.75e^{0.1(2)} = 95.80 - S_T$
$S_T - 95 - (S_T - 90) + 4.75e^{0.1(2)} = 0.80$

Our initial cost is zero. However, our payoff is always non-negative. So we never lose money. This is clearly an arbitrage.

It's important that the two calls are European options. If they are American, they can be exercised at different dates. Hence the following non-arbitrage conditions work only for European options:

$0 \leq C_{Eur}(K_1, T) - C_{Eur}(K_2, T) \leq PV(K_2 - K_1)$ if $K_1 < K_2$
$0 \leq P_{Eur}(K_2, T) - P_{Eur}(K_1, T) \leq PV(K_2 - K_1)$ if $K_1 < K_2$

c.

We are given the following 3 calls:

Strike	$K_1 = 90$	$K_2 = 100$	$K_3 = 105$
Call premium	15	10	6

$\lambda 90 + (1-\lambda) 105 = 100 \qquad \lambda = \frac{1}{3}$

$\rightarrow \frac{1}{3}(90) + \frac{2}{3}(105) = 100$

$C\left[\frac{1}{3}(90) + \frac{2}{3}(105)\right] = C(100) = 10$

$\frac{1}{3}C(90) + \frac{2}{3}C(105) = \frac{1}{3}(15) + \frac{2}{3}(6) = 9$

$C\left[\frac{1}{3}(90) + \frac{2}{3}(105)\right] > \frac{1}{3}C(90) + \frac{2}{3}C(105)$

Hence arbitrage opportunities exist. To arbitrage, we buy low and sell high.

The cheap asset is the diversified portfolio consisting of $\frac{1}{3}$ unit of 90-strike call and $\frac{2}{3}$ unit of 105-strike call.

The expensive asset is the 100-strike call.

Since we can't buy a partial option, we'll buy 3 units of the portfolio (i.e. buy one 90-strike call and two 105-strike calls). Simultaneously,we sell three 100-strike calls.

The payoff at expiration T:

		T	T	T	T
Transaction	$t = 0$	$S_T < 90$	$90 \leq S_T < 100$	$100 \leq S_T < 105$	$S_T \geq 105$
buy 3 portfolios					
buy one 90-strike call	-15	0	$S_T - 90$	$S_T - 90$	$S_T - 90$
buy two 105-strike calls	$2(-6)$	0	0	0	$2(S_T - 105)$
Portfolios total	-27	0	$S_T - 90$	$S_T - 90$	$3S_T - 300$
Sell three 100-strike calls	$3(10)$	0	0	$-3(S_T - 100)$	$-3(S_T - 100)$
Total	3	0	$S_T - 90 \geq 0$	$2(105 - S_T) > 0$	0

$-15 + 2(-6) = -27$
$S_T - 90 + 2(S_T - 105) = 3S_T - 300$
$-27 + 3(10) = 3$
$S_T - 90 - 3(S_T - 100) = 210 - 2S_T = 2(105 - S_T)$
$3S_T - 300 - 3(S_T - 100) = 0$

So we receive \$3 at $t = 0$, but we incur no negative payoff at T. So we'll make at least \$3 free money.

Problem 9.13.

a. If the stock pays dividend, then early exercise of an American call option may be optimal.

Suppose the stock pays dividend at t_D.

Time	0	...	...	t_D	...	...	T

Pro and con for exercising the call early at t_D.

- +. If you exercise the call immediately before t_D, you'll receive dividend and earn interest during $[t_D, T]$
- −. You'll pay the strike price K at t_D, losing interest you could have earned during $[t_D, T]$. If the interest rate, however, is zero, you won't lose any interest.
- −. You throw away the remaining call option during $[t_D, T]$. Had you waited, you would have the call option during $[t_D, T]$

If the accumulated value of the dividend exceeds the value of the remaining call option, then it's optimal to exercise the stock at t_D.

As explained in my study guide, it's never optimal to exercise an American put early if the interest rate is zero.

Problem 9.14.

a. The only reason that early exercise might be optimal is that the underlying asset pays a dividend. If the underlying asset doesn't pay dividend, then it's never optimal to exercise an American call early. Since Apple doesn't pay dividend, it's never optimal to exercise early.

b. The only reason to exercise an American put early is to earn interest on the strike price. The strike price in this example is one share of AOL stock. Since AOL stocks won't pay any dividends, there's no benefit for owning an AOL stock early. Thus it's never optimal to exercise the put.

If the Apple stock price goes to zero and will always stay zero, then there's no benefit for delaying exercising the put; there's no benefit for exercising the put early either (since AOL stocks won't pay dividend). Exercising the put early and exercising the put at maturity have the same value.

If, however, the Apple stock price goes to zero now but may go up in the future, then it's never optimal to exercise the put early. If you don't exercise early, you leave the door open that in the future the Apple stock price may exceed the AOL stock price, in which case you just let your put expire worthless. If the Apple stock price won't exceed the AOL stock price, you can always exercise the put and exchange one Apple stock for one AOL stock. There's no hurry to exercise the put early.

c. If Apple is expected to pay dividend, then it might be optimal to exercise the American call early and exchange one AOL stock for one Apple stock.

However, as long as the AOL stock won't pay any dividend, it's never optimal to exercise the American put early to exchange one Apple stock for one AOL stock.

Problem 9.15.

This is an example where the strike price grows over time.

If the strike price grows over time, the longer-lived option is at least as valuable as the shorter lived option. Refer to Derivatives Markets Page 298.

We have two European calls:

Call #1	$K_1 = 100e^{0.05(1.5)} = 107.788$	$T_1 = 1.5$	$C_1 = 11.50$
Call #2	$K_2 = 100e^{0.05} = 105.127$	$T_2 = 1$	$C_2 = 11.924$

The longer-lived call is cheaper than the shorter-lived call, leading to arbitrage opportunities. To arbitrage, we buy low (Call #1) and sell high (Call #2).

The payoff at expiration $T_1 = 1.5$ if $S_{T_2} < 100e^{0.05} = 105.127$

			T_1	T_1
Transaction	$t = 0$	T_2	$S_{T_1} < 100e^{0.05(1.5)}$	$S_{T_1} \geq 100e^{0.05(1.5)}$
Sell Call #2	11.924	0	0	0
buy Call #1	-11.50		0	$S_{T_1} - 100e^{0.05(1.5)}$
Total	0.424		0	$S_{T_1} - 100e^{0.05(1.5)} \geq 0$

We receive \$0.424 at $t = 0$, yet our payoff at T_1 is always non-negative. This is clearly an arbitrage.

The payoff at expiration $T_1 = 1.5$ if $S_{T_2} \geq 100e^{0.05} = 105.127$

			T_1	T_1
Transaction	$t = 0$	T_2	$S_{T_1} < 100e^{0.05(1.5)}$	$S_{T_1} \geq 100e^{0.05(1.5)}$
Sell Call #2	11.924	$100e^{0.05} - S_{T_2}$	$100e^{0.05(1.5)} - S_{T_1}$	$100e^{0.05(1.5)} - S_{T_1}$
buy Call #1	-11.50		0	$S_{T_1} - 100e^{0.05(1.5)}$
Total	0.424		$100e^{0.05(1.5)} - S_{T_1} < 0$	0

If $S_{T_2} \geq 100e^{0.05}$, then payoff of the sold Call #2 at T_2 is $100e^{0.05} - S_{T_2}$. From T_2 to T_1,

- $100e^{0.05}$ grows into $\left(100e^{0.05}\right) e^{0.05(T_1-T_2)} = \left(100e^{0.05}\right) e^{0.05(0.5)} = 100e^{0.05(1.5)}$
- S_{T_2} becomes S_{T_1} (i.e. the stock price changes from S_{T_2} to S_{T_1})

We receive \$0.424 at $t = 0$, yet our payoff at T_1 can be negative. This is not an arbitrage.

So as long as $S_{T_2} < 100e^{0.05} = 105.127$, there'll be arbitrage opportunities.

Problem 9.16.

Suppose we do the following at $t = 0$:

1. Pay C^a to buy a call
2. Lend $PV(K) = Ke^{-r_L}$ at r_L
3. Sell a put, receiving P^b
4. Short sell one stock, receiving S_0^b

The net cost is $P^b + S_0^b - (C^a + Ke^{-r_L})$.
The payoff at T is:

		If $S_T < K$	If $S_T \geq K$
Transactions	$t = 0$		
Buy a call	$-C^a$	0	$S_T - K$
Lend Ke^{-r_L} at r_L	$-Ke^{r_L}$	K	K
Sell a put	P^b	$S_T - K$	0
Short sell one stock	S_0^b	$-S_T$	$-S_T$
Total	$P^b + S_0^b - (C^a + Ke^{-r_L})$	0	0

The payoff is always zero. To avoid arbitrage, we need to have
$P^b + S_0^b - (C^a + Ke^{-r_L}) \leq 0$

Similarly, we can do the following at $t = 0$:

1. Sell a call, receiving C^b
2. Borrow $PV(K) = Ke^{-r_B}$ at r_B
3. Buy a put, paying P^a
4. Buy one stock, paying S_0^a

The net cost is $\left(C^b + Ke^{-r_B}\right) - \left(P^b + S_0^b\right)$.
The payoff at T is:

		If $S_T < K$	If $S_T \geq K$
Transactions	$t = 0$		
Sell a call	C^b	0	$K - S_T$
Borrow Ke^{-r_B} at r_B	Ke^{-r_B}	$-K$	$-K$
Buy a put	$-P^a$	$K- S_T$	0
Buy one stock	$-S_0^a$	S_T	S_T
Total	$\left(C^b + Ke^{-r_B}\right) - \left(P^b + S_0^b\right)$	0	0

The payoff is always zero. To avoid arbitrage, we need to have
$\left(C^b + Ke^{-r_B}\right) - \left(P^b + S_0^b\right) \leq 0$

Problem 9.17.

a. According to the put-call parity, the payoff of the following position is always zero:

1. Buy the call
2. Sell the put
3. Short the stock
4. Lend the present value of the strike price plus dividend

The existence of the bid-ask spread and the borrowing-lending rate difference doesn't change the zero payoff of the above position. The above position always has a zero payoff whether there's a bid-ask spread or a difference between the borrowing rate and the lending rate.

If there is no transaction cost such as a bid-ask spread, the initial gain of the above position is zero. However, if there is a bid-ask spread, then to avoid arbitrage, the initial gain of the above position should be zero or negative.

The initial gain of the position is:
$\left(P^b + S_0^b\right) - \left[C^a + PV_{r_L}\left(K\right) + PV_{r_L}\left(Div\right)\right]$
There's no arbitrage if
$\left(P^b + S_0^b\right) - \left[C^a + PV_{r_L}\left(K\right) + PV_{r_L}\left(Div\right)\right] \leq 0$

In this problem, we are given

- $r_L = 0.019$
- $r_B = 0.02$
- $S_0^b = 84.85$. We are told to ignore the transaction cost. In addition, we are given that the current stock price is 84.85. So $S_0^b = 84.85$.
- The dividend is 0.18 on November 8, 2004.

To find the expiration date, you need to know this detail. Puts and calls are called equity options at the Chicago Board of Exchange (CBOE). In CBOE, the expiration date of an equity option is the Saturday immediately following the third Friday of the expiration month. (To verify this, go to www.cboe.com. Click on "Products" and read "Production Specifications.")

If the expiration month is November, 2004, the third Friday is November 19. Then the expiration date is November 20.

$T = \frac{11/20/2004 - 10/15/2004}{365} = \frac{36}{365} = 0.09863$

If the expiration month is January, 2005, the third Friday is January 21. Then the expiration date is January 22, 2005.

$T = \frac{1/22/2005 - 10/15/2004}{365}$

Calculate the days between 1/22/2005 and10/15/2004 isn't easy. Fortunately,we can use a calculator. BA II Plus and BA II Plus Professional have "Date" Worksheet. When using Date Worksheet, use the ACT mode. ACT mode calculates the actual days between two dates. If you use the 360 day mode, you are assuming that there are 360 days between two dates.

When using the date worksheet, set DT1 (i.e. Date 1) as October 10, 2004 by entering 10.1504; set DT2 (i.e. Date 2) as January 22, 2004 by entering 1.2204. The calculator should tell you that DBD=99 (i.e. the days between two days is 99 days).

So $T = \frac{1/22/2005 - 10/15/2004}{365} = \frac{99}{365} = 0.271\,23$

If you have trouble using the date worksheet, refer to the guidebook of BA II Plus or BA II Plus Professional.

The dividend day is $t_D = \frac{11/8/2004 - 10/15/2004}{365} = \frac{24}{365} = 0.06\,575$

$PV_{r_L}(Div) = 0.18e^{-0.06575(0.019)} = 0.18$
$PV_{r_L}(K) = Ke^{-0.019T}$
$\left(P^b + S_0^b\right) - \left[C^a + PV_{r_L}(K) + PV_{r_L}(Div)\right]$
$= P^b + 84.85 - \left(C^a + Ke^{-0.019T} + 0.18\right)$

K	T	C^a	P^b	$\left(P^b + S_0^b\right) - \left[C^a + PV_{r_L}(K) + PV_{r_L}(Div)\right]$
75	0.0986	10.3	0.2	$0.2 + 84.85 - \left(10.3 + 75e^{-0.019\times0.0986} + 0.18\right) = -0.29$
80	0.0986	5.6	0.6	$0.6 + 84.85 - \left(5.6 + 80e^{-0.019\times0.0986} + 0.18\right) = -0.18$
85	0.0986	2.1	2.1	$2.1 + 84.85 - \left(2.1 + 85e^{-0.019\times0.0986} + 0.18\right) = -0.17$
90	0.0986	0.35	5.5	$5.5 + 84.85 - \left(0.35 + 90e^{-0.019\times0.0986} + 0.18\right) = -1.15$
75	0.271 2	10.9	0.7	$0.7 + 84.85 - \left(10.9 + 75e^{-0.019\times0.2712} + 0.18\right) = -0.14$
80	0.271 2	6.7	1.45	$1.45 + 84.85 - \left(6.7 + 80e^{-0.019\times0.2712} + 0.18\right) = -0.17$
85	0.271 2	3.4	3.1	$3.1 + 84.85 - \left(3.4 + 85e^{-0.019\times0.2712} + 0.18\right) = -0.19$
90	0.271 2	1.35	6.1	$6.1 + 84.85 - \left(1.35 + 90e^{-0.019\times0.2712} + 0.18\right) = -0.12$

b. According to the put-call parity, the payoff of the following position is always zero:

1. Sell the call
2. Borrow the present value of the strike price plus dividend
3. Buy the put
4. Buy one stock

If there is transaction cost such as the bid-ask spread, then to avoid arbitrage, the initial gain of the above position is zero. However, if there is a bid-ask spread, the initial gain of the above position can be zero or negative.

The initial gain of the position is:
$C^b + PV_{r_B}(K) + PV_{r_B}(Div) - (P^a + S_0^a)$
There's no arbitrage if
$C^b + PV_{r_B}(K) + PV_{r_B}(Div) - (P^a + S_0^a) \leq 0$
$PV_{r_B}(Div) = 0.18e^{-0.06575(0.02)} = 0.18$
$PV_{r_L}(K) = Ke^{-0.02T}$
$C^b + PV_{r_B}(K) + PV_{r_B}(Div) - (P^a + S_0^a) = C^b + Ke^{-0.02T} + 0.18 - (P^a + 84.85)$

K	T	C^b	P^a	$C^b + PV_{r_B}(K) + PV_{r_B}(Div) - (P^a + S_0^a)$
75	0.0986	9.9	0.25	$9.9 + 75e^{-0.02\times0.0986} + 0.18 - (0.25 + 84.85) = -0.17$
80	0.0986	5.3	0.7	$5.3 + 80e^{-0.02\times0.0986} + 0.18 - (0.7 + 84.85) = -0.23$
85	0.0986	1.9	2.3	$1.9 + 85e^{-0.02\times0.0986} + 0.18 - (2.3 + 84.85) = -0.24$
90	0.0986	0.35	5.8	$0.35 + 90e^{-0.02\times0.0986} + 0.18 - (5.8 + 84.85) = -0.30$
75	0.271 2	10.5	0.8	$10.5 + 75e^{-0.02\times0.2712} + 0.18 - (0.8 + 84.85) = -0.38$
80	0.271 2	6.5	1.6	$6.5 + 80e^{-0.02\times0.2712} + 0.18 - (1.6 + 84.85) = -0.20$
85	0.271 2	3.2	3.3	$3.2 + 85e^{-0.02\times0.2712} + 0.18 - (3.3 + 84.85) = -0.23$
90	0.271 2	1.2	6.3	$1.2 + 90e^{-0.02\times0.2712} + 0.18 - (6.3 + 84.85) = -0.26$

Problem 9.18.

Suppose there are 3 options otherwise identical but with different strike price $K_1 < K_2 < K_3$ where $K_2 = \lambda K_1 + (1-\lambda) K_2$ and $0 < \lambda < 1$.

Then the price of the middle strike price K_2 must not exceed the price of a diversified portfolio consisting of λ units of K_1-strike option and $(1-\lambda)$ units of K_2-strike option:

$$C[\lambda K_1 + (1-\lambda) K_3] \leq \lambda C(K_1) + (1-\lambda) C(K_3)$$
$$P[\lambda K_1 + (1-\lambda) K_3] \leq \lambda P(K_1) + (1-\lambda) P(K_3)$$

The above conditions are called the convexity of the option price with respect to the strike price. They are equivalent to the textbook Equation 9.17 and 9.18.

If the above conditions are violated, arbitrage opportunities exist.

K	T	C^b	C^a
80	0.271 2	6.5	6.7
85	0.271 2	3.2	3.4
90	0.271 2	1.2	1.35

$85 = \lambda (80) + (1 - \lambda)(90) \qquad \rightarrow \lambda = 0.5$

a.

If we buy a 80-strike call, buy a 90-strike call, sell two 85-strike calls

- A 80-strike call and a 90-strike call form a diversified portfolio of calls, which is always as good as two 85-strike calls
- So the cost of buying a 80-strike call and a 90-strike call can never be less than the revenue of selling two 85-strike calls

What we pay if we buy a 80-strike call and a 90-strike call: $6.7 + 1.35 = 8.05$

What we get if we sell two 85-strike calls: $3.2 \times 2 = 6.4$

$8.05 > 6.4$

So the convexity condition is met.

I recommend that you don't bother memorizing textbook Equation 9.17 and 9.18.

b. If we sell a 80-strike call, sell a 90-strike call, buy two 85-strike calls

- A 80-strike call and a 90-strike call form a diversified portfolio of calls, which is always as good as two 85-strike calls
- So the revenue of selling a 80-strike call and a 90-strike call should never be less than the cost of buying two 85-strike calls.

What we get if we sell a 80-strike call and a 90-strike call: $6.5 + 1.2 = 7.7$

What we pay if we buy two 85-strike calls: $3.4 \times 2 = 6.8$

$7.7 > 6.8$

So the convexity condition is met.

c. To avoid arbitrage, the following two conditions must be met:

$C[\lambda K_1 + (1-\lambda) K_3] \leq \lambda C(K_1) + (1-\lambda) C(K_3)$

$P[\lambda K_1 + (1-\lambda) K_3] \leq \lambda P(K_1) + (1-\lambda) P(K_3)$

These conditions must be met no matter you are a market-maker or anyone else buying or selling options, no matter you pay a bid-ask spread or not.