Deeper Understanding, Faster Calc: SOA MFE and CAS Exam 3F

Yufeng Guo

Contents

Introduction

This study guide is for SOA MFE and CAS Exam 3F. Before you start, make sure you have the following items:

1. Derivatives Markets, the 2nd edition.
2. Errata of Derivatives Markets. You can download the errata at `http://www.kellogg.northwestern.edu/faculty/mcdonald/htm/typos2e_01.html`. Don't miss the errata about the textbook pages 780 through 788.
3. Download the syllabus from the SOA or CAS website.
4. Download the sample MFE problems and solutions from the SOA website.
5. Download the recent SOA MFE and CAS Exam 3 problems.

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Chapter 12

Black-Scholes

You probably have memorized the famous Black-Scholes call and put price formulas and can readily calculate the price of a plain vanilla European call or put option. But what if SOA throws a tricky derivative at you? Here are a few examples of ad hoc contracts:

- An option allows you to pay K and receive $\ln S_T$ at T. What's its price?
- An option allows you to pay K and receive $\sqrt{S_T}$ at T. What's its price?
- An option pays $(S_t - K)^2$ at T only if $S_T > K$. What's its price?

To tackle non-standard derivatives, you need to do more than memorize the Black-Scholes formula. In this chapter, you'll learn how to derive the Black-Scholes formula from the ground up and how to price an ad hoc contract.

The math behind the Black-Scholes formula is simple. All you need to know is (1) some Exam P level calculus, and (2) the risk neutral pricing (an option is worth its expected payoff discounted at the risk free rate).

First, though, let's review the basics of the Black-Scholes formula.

12.1 Introduction to the Black-Scholes formula

12.1.1 Call and put option price

The price of a European call option is:

$$C(S, K, \sigma, r, T, \delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \tag{12.1}$$

The price of a European put option is:

$$P(S, K, \sigma, r, T, \delta) = -Se^{-\delta T} N(-d_1) + Ke^{-rT} N(-d_2) \tag{12.2}$$

$$d_1 = \frac{\ln \frac{S}{K} + \left(r - \delta + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \tag{12.3}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{12.4}$$

Notations used in Equation 12.1, 12.3, and 12.4:

- S, the current stock price (i.e. the stock price when the option is written)
- K, the strike price
- r, the continuously compounded risk-free interest rate per year
- δ, the continuously compounded dividend rate per year
- σ, the annualized standard deviation of the continuously compounded stock return (i.e. stock volatility)
- T, option expiration time
- $N(d) = P(z \leq d)$ where z is a standard normal random variable
- $C(S, K, \sigma, r, T, \delta)$, the price of a European call option with parameters $(S, K, \sigma, r, T, \delta)$
- $P(S, K, \sigma, r, T, \delta)$, the price of a European put option with parameters $(S, K, \sigma, r, T, \delta)$

Tip 12.1.1. *To help memorize Equation 12.2, we can rewrite Equation 12.2 similar to Equation 12.1 as* $P(S, K, \sigma, r, T, \delta) = (-S) e^{-\delta T} N(-d_1) - (-K) e^{-rT} N(-d_2)$. *In other words, change* S, K, d_1, *and* d_2 *in Equation 12.1 and you'll get Equation 12.2.*

Example 12.1.1. *Reproduce the textbook example 12.1. This is the recap of the information.* $S = 41$, $K = 40$,$r = 0.08$, $\sigma = 0.3$, $T = 0.25$ *(i.e. 3 months), and* $\delta = 0$. *Calculate the price of the price of a European call option.*

$$d_1 = \frac{\ln \frac{S}{K} + \left(r - \delta + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}$$

$$= \frac{\ln \frac{41}{40} + \left(0.08 - 0 + \frac{1}{2} \times 0.3^2\right) 0.25}{0.3\sqrt{0.25}} = 0.3730$$

$d_2 = d_1 - \sigma\sqrt{T} = 0.3730 - 0.3\sqrt{0.25} = 0.2230$

$N(d_1) = 0.645\,4 \qquad N(d_2) = 0.588\,2$

$C = 41e^{-0(0.25)}0.645\,4 - 40e^{-0.08(0.25)}0.588\,2 = 3.\,399$

Example 12.1.2. *Reproduce the textbook example 12.2. This is the recap of the information.* $S = 41$, $K = 40$,$r = 0.08$, $\sigma = 0.3$, $T = 0.25$ *(i.e. 3 months), and* $\delta = 0$. *Calculate the price of the price of a European put option.*

$N(-d_1) = 1 - N(d_1) = 1 - 0.6454 = 0.3546$
$N(-d_2) = 1 - N(d_2) = 1 - 0.5882 = 0.4118$
$P = -41e^{-0(0.25)}0.3546 + 40e^{-0.08(0.25)}0.4118 = 1.607$

12.1.2 When is the Black-Scholes formula valid?

Assumptions under the Black-Scholes formula:

Assumptions about the distribution of stock price:

1. Continuously compounded returns on the stock are normally distributed (i.e. stock price is lognormally distributed) and independent over time
2. The volatility of the continuously compounded returns is known and constant
3. Future dividends are known, either as a dollar amount (i.e. D and T_D are known in advance) or as a fixed dividend yield (i.e. δ is a known constant)

Assumptions about the economic environment

1. The risk-free rate is known and fixed (i.e. r is a known constant)
2. There are no transaction costs or taxes
3. It's possible to short-sell costlessly and to borrow at the risk-free rate

12.2 Derive the Black-Scholes formula

By learning how to derive the Black-Scholes formula, we'll be able to remove the black-box behind the formula and recreate the formula instantly. First, some basics.

What's the density function of a standard normal random variable?
If $z \sim N(0,1)$, then $f(z) = \frac{1}{\sqrt{2\pi}}e^{-0.5z^2}$, $\Phi(a) = P(z \leq a) = \int_{-\infty}^{a} f(z)\,dz$

How can we convert a normal variable to a standard normal variable?
If $Z \sim N(\mu, \sigma^2)$, then set $z = \frac{Z-\mu}{\sigma}$; $Z = \mu + z\sigma$

What's the stock price S_t under the Black-Scholes assumption in the real world?
DM 20.13: $S_t = S_0 \exp\left[\left(\alpha - \delta - 0.5\sigma^2\right)t + \sigma\sqrt{t}z\right]$, $z \sim N(0,1)$

What's S_t under the Black-Scholes assumption in the risk neutral world Q?
Set $\alpha = r$. Then $S_t \overset{Q}{=} S_0 \exp\left[\left(r - \delta - 0.5\sigma^2\right)t + \sigma\sqrt{t}z\right]$, $z \sim N(0,1)$

Next, let's derive some normal random variable related integral shortcuts. The first shortcut. For a standard normal random variable $z \sim N(0,1)$ and a constant d,

$$\int_d^{\infty} f(z)\,dz = P(z > d) = \Phi(-d) \tag{12.5}$$

Proof. Clearly, $\int_d^{\infty} f(z)\,dz = P(z > d)$. We just need to prove that $P(z > d) = \Phi(-d)$.

$P(z > d) = 1 - P(z < d) = 1 - \Phi(d)$.

Notice that $P(z = d) = 0$ (the probability for a continuous random variable to take on a single value is zero).

Hence $P(z > d) = 1 - P(z < d) - P(z = d) = 1 - P(z < d)$. Then using the formula $\Phi(d) + \Phi(-d) = 1$, we get 12.5.

The second integral shortcut:

$$\int e^{\sigma z} f(z)\,dz = e^{-0.5\sigma^2} \int \frac{1}{\sqrt{2\pi}} e^{-0.5(z-\sigma)^2} dz \tag{12.6}$$

Proof. $z \sim N(0,1)$ and $f(z) = \frac{1}{\sqrt{2\pi}} e^{-0.5z^2}$

$\int e^{\sigma z} f(z)\,dz = \int e^{\sigma z} \frac{1}{\sqrt{2\pi}} e^{-0.5z^2} dz = \int \frac{1}{\sqrt{2\pi}} e^{-0.5z^2+\sigma z} dz$

$-0.5z^2 + \sigma z = -0.5\left(z^2 - 2\sigma z + \sigma^2\right) + 0.5\sigma^2 = -0.5(z-\sigma)^2 + 0.5\sigma^2$

$\int \frac{1}{\sqrt{2\pi}} e^{-0.5z^2+\sigma z} dz = \int \frac{1}{\sqrt{2\pi}} e^{-0.5(z-\sigma)^2+0.5\sigma^2} dz = e^{0.5\sigma^2} \int \frac{1}{\sqrt{2\pi}} e^{-0.5(z-\sigma)^2} dz$

The third shortcut is:

$$\int_d^{\infty} e^{\sigma z} f(z)\,dz = e^{0.5\sigma^2} \Phi(\sigma - d) \tag{12.7}$$

Proof. $\int_d^{\infty} e^{\sigma z} f(z)\,dz = e^{-0.5\sigma^2} \int_d^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5(z-\sigma)^2} dz$

Set $t = z - \sigma$. $\quad \int_d^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5(z-\sigma)^2} dz = \int_{d-\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5t^2} dt$

$\frac{1}{\sqrt{2\pi}} e^{-0.5t^2}$ is the density of $t \sim N(0,1)$

$\rightarrow \int_{d-\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5t^2} dt = \Phi(-(d-\sigma)) = \Phi(\sigma - d)$.

$\rightarrow \int_d^{\infty} e^{\sigma z} f(z)\,dz = e^{0.5\sigma^2} \Phi(\sigma - d)$.

Problem 12.1.

For a normal random variable $X \sim N\left(\mu, \sigma^2\right)$, calculate $E\left(e^X\right)$.

Solution.

$X = \mu + \sigma z, \qquad z \sim N(0,1).$

$E\left(e^X\right) = \int_{-\infty}^{\infty} e^{\mu+\sigma z} f(z)\, dz = e^{\mu} \int_{-\infty}^{\infty} e^{\sigma z} f(z)\, dz$
Use Equation $\int_d^{\infty} e^{\sigma z} f(z)\, dz = e^{0.5\sigma^2} \Phi(\sigma - d)$, set $d = -\infty$:
$\int_{-\infty}^{\infty} e^{\sigma z} f(z)\, dz = e^{0.5\sigma^2} \Phi(\sigma - -\infty) = e^{0.5\sigma^2} \Phi(\infty) = e^{0.5\sigma^2} \times 1 = e^{0.5\sigma^2}$

$\Longrightarrow \qquad E\left(e^X\right) = e^{\mu} e^{0.5\sigma^2} = e^{\mu+0.5\sigma^2} = e^{E(X)+0.5Var(X)}$

$$E\left(e^{X \sim N\left(\mu,\sigma^2\right)}\right) = e^{E(X)+0.5Var(X)} = e^{\mu+0.5\sigma^2} \tag{12.8}$$

Problem 12.2.

For a normal random variable $X \sim N\left(\mu, \sigma^2\right)$, calculate $\int_d^{\infty} e^X f(z)\, dz$ where $z \sim N(0,1)$ and $f(z) = \frac{1}{\sqrt{2\pi}} e^{-0.5z^2}$.

Solution.

$\int_d^{\infty} e^{x(z)} f(z)\, dz = \int_d^{\infty} e^{\mu+\sigma z} f(z)\, dz = e^{\mu} \int_d^{\infty} e^{\sigma z} f(z)\, dz = e^{\mu} e^{0.5\sigma^2} \Phi(\sigma - d) =$
$e^{\mu+0.5\sigma^2} \Phi(\sigma - d) = E\left(e^X\right) \Phi(\sigma - d)$

$$\int_d^{\infty} e^X f(z)\, dz = E\left(e^X\right) \Phi(\text{std dev of } X - d) = E\left(e^X\right) \Phi(\sigma - d) \tag{12.9}$$

Tip 12.2.1. *Whenever you need to calculate* $\int_d^{\infty} e^X f(z)\, dz$ *where* $X \sim N\left(\mu, \sigma^2\right)$, *first calculate* $E\left(e^X\right) = e^{\mu+0.5\sigma^2}$. *Next, multiply* $E\left(e^X\right)$ *by* $\Phi(\sigma - d)$.

Problem 12.3.

For a normal random variable $X \sim N\left(\mu, \sigma^2\right)$, calculate $\int_{-\infty}^{d} e^X f(z)\, dz$ where $z \sim N(0,1)$ and $f(z) = \frac{1}{\sqrt{2\pi}} e^{-0.5z^2}$.

Solution.

$\int_{-\infty}^{d} e^X f(z)\,dz = \int_{-\infty}^{\infty} e^X f(z)\,dz - \int_{d}^{\infty} e^X f(z)\,dz = E\left(e^X\right) - E\left(e^X\right)\Phi(\sigma - d) = E\left(e^X\right)[1 - \Phi(\sigma - d)] = E\left(e^X\right)\Phi[-(\sigma - d)]$

$$\int_{-\infty}^{d} e^X f(z)\,dz = E\left(e^X\right)\Phi[-(\text{std dev of } X - d)] = E\left(e^X\right)\Phi[-(\sigma - d)] \tag{12.10}$$

Tip 12.2.2. *Whenever you need to calculate* $\int_{-\infty}^{d} e^X f(z)\,dz$ *where* $X \sim N\left(\mu, \sigma^2\right)$, *first calculate* $E\left(e^X\right) = e^{\mu + 0.5\sigma^2}$. *Next, multiply* $E\left(e^X\right)$ *by* $\Phi[-(\sigma - d)]$.

Problem 12.4.

Derive the Black-Scholes call option formula 12.1.

Solution.

Consider two contracts:

- Contract #1 pays S_T at T if $S_T > K$. The payoff at T is $X_T^1 = \begin{cases} S_T & \text{If } S_T > K \\ 0 & \text{If } S_T \leq K \end{cases}$.
- Contract #2 pays K at T if $S_T > K$. The payoff at T is $X_T^2 = \begin{cases} K & \text{If } S_T > K \\ 0 & \text{If } S_T \leq K \end{cases}$

Let V_1 and V_2 represent the price of the Contract #1 and #2 respectively. Under the risk neutral pricing, the price of a contract is just the expected payoff discounted at the risk free rate. Hence $V_1 = e^{-rT}E^Q\left(X_T^1\right)$ and $V_2 = e^{-rT}E^Q\left(X_T^2\right)$, where Q is the risk-neutral world.

Since $X_T^1 - X_T^2 = \begin{cases} S_T - K & \text{If } S_T > K \\ 0 & \text{If } S_T \leq K \end{cases}$ is the payoff of a call option, the call option price is equal to $V = V_1 - V_2$.

$S_T \overset{Q}{=} S_0 e^X, \qquad X \overset{Q}{\sim} N\left[\mu = \left(r - \delta - 0.5\sigma^2\right)T, Var = \sigma^2 T\right]$

Solve $S_T(z) = S_0 \exp\left[\left(r - \delta - 0.5\sigma^2\right)T + \sigma\sqrt{T}z\right] > K$

$$\rightarrow \qquad z > \frac{\ln\dfrac{K}{S_0} - \left(r - \delta - 0.5\sigma^2\right)T}{\sigma\sqrt{T}} = -d_2$$

Notice $d_2 = \dfrac{\ln\dfrac{S_0}{K} + \left(r - \delta - 0.5\sigma^2\right)T}{\sigma\sqrt{T}}$ in the Black-Scholes formula. Then $S_T(z) > K$ is the same as $z > -d_2$. Hence we can write X_T^1 and X_T^2 as follows:

$$X_T^1 = \begin{cases} S_T & \text{If } S_T > K \\ 0 & \text{If } S_T \le K \end{cases} = \begin{cases} S_T & \text{If } z > -d_2 \\ 0 & \text{If } z \le -d_2 \end{cases}$$

$$X_T^2 = \begin{cases} K & \text{If } S_T > K \\ 0 & \text{If } S_T \le K \end{cases} = \begin{cases} K & \text{If } z > -d_2 \\ 0 & \text{If } z \le -d_2 \end{cases}$$

$E^Q\left(X_T^2\right) = \int_{-\infty}^{-d_2} 0f(z)\,dz + \int_{-d_2}^{\infty} Kf(z)\,dz = \int_{-d_2}^{\infty} Kf(z)\,dz = K\int_{-d_2}^{\infty} f(z)\,dz = K\Phi(d_2)$

$\Longrightarrow \qquad V_2 = e^{-rT}E^Q(Y) = Ke^{-rT}\Phi(d_2)$

Notice $\Phi(d_2) = \int_{-d_2}^{\infty} f(z) = P^Q(S_T > K)$ is the risk neutral probability of $S_T > K$.

$E^Q\left(X_T^1\right) = \int\limits_{S_T>K} S_T(z)\,f(z)\,dx + \int\limits_{S_T<K} 0f(z)\,dz = \int\limits_{S_T>K} S_T(z)\,f(z)\,dz = \int_{-d_2}^{\infty} S_T(z)\,f(z)\,dz = S_0\int_{-d_2}^{\infty} e^{X(z)}f(z)\,dz = S_0\int_{-d_2}^{\infty} e^{\left(r-\delta-0.5\sigma^2\right)T+\sigma\sqrt{T}z}f(z)\,dz$

Use the formula: $\int_d^{\infty} e^X f(z)\,dz = E\left(e^X\right)\Phi(\text{std dev of } X - d)$

$\Longrightarrow \qquad S_0\int_{-d_2}^{\infty} e^X f(z)\,dz = S_0E\left(e^X\right)\Phi(\text{std dev of } X + d_2)$

$E\left(e^X\right) = e^{E(X)+0.5Var(X)} = e^{\left(r-\delta-0.5\sigma^2\right)T+0.5\sigma^2T} = e^{(r-\delta)T}$

$\Phi(\text{std dev of } X + d_2) = \Phi\left(\sigma\sqrt{T} + d_2\right) = \Phi(d_1)$

$\Longrightarrow \qquad E^Q\left(X_T^1\right) = S_0e^{(r-\delta)T}\Phi(d_1)$

$\Longrightarrow \qquad V = V_1 - V_2 = S_0e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$

Problem 12.5.

Derive the Black-Scholes put option formula 12.2.

Solution.

Consider two contracts:

- Contract #1 pays S_T at T if $S_T < K$. The payoff at T is $Y_T^1 = \begin{cases} S_T & \text{If } S_T < K \\ 0 & \text{If } S_T \ge K \end{cases}$.
- Contract #2 pays K at T if $S_T < K$. The payoff at T is $Y_T^2 = \begin{cases} K & \text{If } S_T < K \\ 0 & \text{If } S_T \ge K \end{cases}$

Let V_1 and V_2 represent the price of the Contract #1 and #2 respectively. $V_1 = e^{-rT}E^Q\left(Y_T^1\right)$ and $V_2 = e^{-rT}E^Q\left(Y_T^2\right)$, where Q is the risk-neutral world. Since $Y_T^2 - Y_T^1 = \begin{cases} K - S_T & \text{If } S_T < K \\ 0 & \text{If } S_T \ge K \end{cases}$ is the payoff of a put option, the put option price is equal to $V = V_2 - V_1$.

Solve $S_T(z) > K$.

$$\rightarrow \qquad z < \frac{\ln \frac{K}{S_0} - \left(r - \delta - 0.5\sigma^2\right)T}{\sigma\sqrt{T}} = -d_2$$

Then $S_T(z) < K$ is the same as $z < -d_2$. Hence we write X_T^1 and X_T^2 as follows:

$$Y_T^1 = \begin{cases} S_T & \text{If } S_T < K \\ 0 & \text{If } S_T \geq K \end{cases} = \begin{cases} S_T & \text{If } z < -d_2 \\ 0 & \text{If } z \leq -d_2 \end{cases}$$

$$Y_T^2 = \begin{cases} K & \text{If } S_T < K \\ 0 & \text{If } S_T \geq K \end{cases} = \begin{cases} K & \text{If } z < -d_2 \\ 0 & \text{If } z \geq -d_2 \end{cases}$$

$E^Q\left(Y_T^2\right) = \int_{-\infty}^{-d_2} Kf(z)\,dz + \int_{-d_2}^{\infty} 0f(z)\,dz = \int_{-\infty}^{-d_2} Kf(z)\,dz = K\Phi(-d_2)$
$\Longrightarrow \qquad V_2 = e^{-rT}E^Q(Y) = Ke^{-rT}\Phi(-d_2)$

$\Phi(-d_2) = \int_{-\infty}^{-d_2} f(z)\,dz = P^Q(S_T < K)$ is the risk neutral probability of $S_T < K$.

$E^Q\left(Y_T^1\right) = \int_{-\infty}^{-d_2} S_T(z) f(z)\,dz + \int_{-d_2}^{\infty} 0f(z)\,dz = \int_{-\infty}^{-d_2} S_T(z) f(z)\,dz = \int_{-\infty}^{-d_2} S_0 e^X f(z)\,dz = S_0 \int_{-\infty}^{-d_2} e^X f(z)\,dz$

where $X \overset{Q}{\sim} N\left[\mu = \left(r - \delta - 0.5\sigma^2\right)T, Var = \sigma^2 T\right]$

$\int_{-\infty}^{-d_2} e^X f(z)\,dz = E\left(e^X\right)\Phi\left[-(\text{std dev of } X - (-d_2))\right] = E\left(e^X\right)\Phi\left[-\left(\sigma\sqrt{T} + d_2\right)\right] = E\left(e^X\right)\Phi(-d_1)$
$E\left(e^X\right) = e^{E(X) + 0.5Var(X)} = e^{\left(r - \delta - 0.5\sigma^2\right)T + 0.5\sigma^2 T} = e^{(r-\delta)T}$

$$\Longrightarrow \qquad E^Q\left(Y_T^1\right) = e^{(r-\delta)T}\Phi(-d_1) \qquad V_2 = e^{-rT}\left[E^Q\left(Y_T^1\right)\right] = e^{-\delta T}\Phi(-d_1)$$

$$\Longrightarrow \qquad V = V_2 - V_1 = e^{-rT}K\Phi(-d_2) - S_0 e^{-\delta T}\Phi(-d_1)$$

Problem 12.6.

What's the meaning of $\Phi(d_2)$ and $\Phi(-d_2)$ in the Black-Scholes formula?

Solution.

$\Phi(d_2) = P^Q(S_T > K)$ is the risk neutral probability of $S_T > K$.
$\Phi(-d_2) = P^Q(S_T < K)$ is the risk neutral probability of $S_T < K$.

Problem 12.7.

Since $\Phi(d_2)$ is the risk neutral probability of $S_T > K$, I thought the call price should be $S_0 e^{-\delta T}\Phi(d_2) - Ke^{-rT}\Phi(d_2)$, but the Black-Scholes formula is $S_0 e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$. Why?

Solution.

The wrong formula $C = S_0 e^{-\delta T}\Phi(d_2) - Ke^{-rT}\Phi(d_2) = \left(S_0 e^{-\delta T} - Ke^{-rT}\right)\Phi(d_2)$ can easily produce a negative or zero call price when $S_0 \leq K$.

For example, set $\delta = r = 0$ and $S_0 = K$. The wrong formula is $C = (S_0 - K)\Phi(d_2) = 0$, but a call price is always positive.

Here's another example. To simplify calculation, set $\delta = 0$, $r = 0.06$, $S_0 = 50$, $K = 100$, $\sigma = 1$, $T = 1$.

$$d_2 = \frac{\ln\frac{S}{K} + \left(r - \delta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\ln\frac{50}{100} + \left(0.06 - 0 - 0.5 \times 1^2\right)1}{1\sqrt{1}} = -1.133\,15$$

$d_1 = d_2 + \sigma\sqrt{T} = -1.133\,15 + 1\sqrt{1} = -0.133\,15$

$N(d_2) = \text{NormalDist}(-1.133\,15) = 0.128\,58$
$N(d_1) = \text{NormalDist}(-0.133\,15) = 0.447\,04$

Notice that $N\left(d_1 = d_2 + \sigma\sqrt{T}\right) > N(d_2)$ because the cumulative density function $N(z)$ is an increasing function of z.

The correct call price is
$C = S_0 e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2) = 50 \times 0.447\,04 - 100e^{-0.06} \times 0.128\,58 = 10.24$

The wrong call price is
$C = S_0 e^{-\delta T}\Phi(d_2) - Ke^{-rT}\Phi(d_2) = \left(50 - 100e^{-0.06}\right)0.128\,58 = -5.68$

Third example. Set $\delta = 0$, $r = 0.06$, $S_0 = 1$, $K = 100$, $\sigma = 1$, $T = 1$

$$d_2 = \frac{\ln\frac{S}{K} + \left(r - \delta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\ln\frac{1}{100} + \left(0.06 - 0 - 0.5 \times 1^2\right)1}{1\sqrt{1}} = -5.045\,17$$

$d_1 = d_2 + \sigma\sqrt{T} = -5.045\,17 + 1\sqrt{1} = -4.045\,17$

$N(d_2) = \text{NormalDist}(-5.045\,17) = 2.265\,59 \times 10^{-7}$
$N(d_1) = \text{NormalDist}(-4.045\,17) = 2.614\,26 \times 10^{-5}$

$$\frac{N(d_1)}{N(d_2)} = \frac{2.614\,26 \times 10^{-5}}{2.265\,59 \times 10^{-7}} = 115.389\,8$$

The correct call price is

$C = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) = \Phi(d_2)\left(S_0\dfrac{\Phi(d_1)}{\Phi(d_2)} - Ke^{-rT}\right) = 2.265\,59 \times 10^{-7}\left(1 \times 115.389\,8 - 100e^{-0.06}\right) = 4.8 \times 10^{-6}$

In this example, as K becomes much greater than S_0, $\dfrac{\Phi(d_1)}{\Phi(d_2)}$ gets big as well, making the call price $\Phi(d_2)\left(S_0\dfrac{\Phi(d_1)}{\Phi(d_2)} - Ke^{-rT}\right)$ positive.

In contrast, the wrong formula $C = \left(S_0 - Ke^{-rT}\right)\Phi(d_2) = \left(1 - 100e^{-0.06}\right)$ $2.265\,59 \times 10^{-7} = -2.11 \times 10^{-5}$ produces a negative call price.

Similarly, though $\Phi(-d_2) = P^Q(S_T < K)$ is the risk neutral probability of $S_T < K$, the put price is **NOT** $V = Ke^{-rT}\Phi(-d_2) - S_0e^{-\delta T}N(-d_2)$ but $V = Ke^{-rT}\Phi(-d_2) - S_0e^{-\delta T}N(-d_1)$ Since $-d_2 > -d_2 = -d_1 - \sigma\sqrt{T}$, we have $\Phi(-d_2) > N(-d_1)$. This makes the put price $Ke^{-rT}\Phi(-d_2) - S_0e^{-\delta T}N(-d_1)$ positive.

Problem 12.8.

Verify that

- the first term of the Black-Scholes call price formula $S_0e^{-\delta T}N(d_1)$ is equal to $e^{-rT}\left[E^Q(S_T|S_T > K)\,E^Q(S_T > K)\right]$, not equal to $e^{-rT}\left[E^Q(S_T)\,E^Q(S_T > K)\right]$
- the 2nd term of the Black-Scholes put price formula $S_0e^{-\delta T}N(-d_1)$ is equal to $e^{-rT}\left[E^Q(S_T|S_T < K)\,E^Q(S_T < K)\right]$, not equal to $e^{-rT}\left[E^Q(S_T)\,E^Q(S_T < K)\right]$

Solution.

$S_T > K$ is the same as $z > -d_2$
Notice that

$$\underbrace{\int_{-\infty}^{\infty} S_T(z)\,f(z)\,dz}_{E^Q(S_T)} = \underbrace{\int_{-d_2}^{\infty} S_T(z)\,f(z)\,dz}_{E^Q\left(X_T^1\right)=E^Q(S_T|S_T>K)P^Q(S_T>K)} + \underbrace{\int_{-\infty}^{-d_2} S_T(z)\,f(z)\,dz}_{E^Q\left(Y_T^1\right)=E^Q(S_T|S_T<K)P^Q(S_T<K)}$$

where X_T^1 and Y_T^1 are the payoffs of the following contracts:

- Contract #1 pays S_T at T if $S_T > K$. The payoff at T is $X_T^1 = \begin{cases} S_T & \text{If } S_T > K \\ 0 & \text{If } S_T < K \end{cases}$.
- Contract #1 pays S_T at T if $S_T < K$. The payoff at T is $Y_T^1 = \begin{cases} S_T & \text{If } S_T < K \\ 0 & \text{If } S_T > K \end{cases}$

$$E^Q\left(X_T^1\right)=E^Q\left(S_T|S_T>K\right)E^Q\left(S_T>K\right)=S_0E^Q\left(e^X\right)N\left(d_1\right)=S_0e^{(r-\delta)T}N\left(d_1\right)$$

$$\Longrightarrow \qquad e^{-rT}\left[E^Q\left(X_T^1\right)\right]=e^{-rT}\left[E^Q\left(S_T|S_T>K\right)E^Q\left(S_T>K\right)\right]=S_0e^{-\delta T}N\left(d_1\right)$$

$$E^Q\left(Y_T^1\right)=E^Q\left(S_T|S_T<K\right)E^Q\left(S_T<K\right)=S_0E^Q\left(e^X\right)N\left(-d_1\right)=S_0e^{(r-\delta)T}N\left(-d_2\right)$$

$$\Longrightarrow \qquad e^{-rT}\left[E^Q\left(Y_T^1\right)\right]=e^{-rT}\left[E^Q\left(S_T|S_T<K\right)E^Q\left(S_T<K\right)\right]=S_0e^{-\delta T}N\left(-d_1\right)$$

Problem 12.9.

What's the meaning of $N(d_1)$ and $N(-d_1)$?

Solution.

Later in the chapter about the V world (where $S_te^{\delta t}$ is used as the numeraire) we'll learn that $N(d_1)=P^V(S_T>K)$ (the V world probability of $S_T>K$) and $N(-d_1)=P^V(S_T<K)$ (the V world probability of $S_T<K$). Now let's interpret $N(d_1)$ and $N(-d_1)$ in a different way.

Consider two contracts:

- Contract #1 pays S_T at T if $S_T>K$. The payoff at T is $X_T^1=\begin{cases} S_T & \text{If } S_T>K \\ 0 & \text{If } S_T<K \end{cases}$.
- Contract #2 pays S_T at T if $S_T<K$. The payoff at T is $Y_T^1=\begin{cases} S_T & \text{If } S_T<K \\ 0 & \text{If } S_T>K \end{cases}$

Notice that

$$E^Q\left(S_T\right)=\int_{-\infty}^{\infty}S_T(z)f(z)\,dz=\underbrace{\int_{-d_2}^{\infty}S_T(z)f(z)\,dz}_{E^Q\left(X_T^1\right)=E^Q\left(S_T\right)\Phi\left(d_1\right)}+\underbrace{\int_{-\infty}^{-d_2}S_T(z)f(z)\,dz}_{E^Q\left(Y_T^1\right)=E^Q\left(S_T\right)\Phi\left(-d_1\right)}$$

$E^Q(S_T)$ consists of two parts, one to pay for $E^Q\left(X_T^1\right)=E^Q(S_T)\Phi(d_1)$ and the other to pay for $E^Q\left(Y_T^1\right)=E^Q(S_T)\Phi(-d_1)$. We see that

- $\Phi(d_1)$ is the fraction of $E^Q(S_T)$ to pay for $E^Q\left(X_T^1\right)$, and
- $\Phi(-d_1)=1-\Phi(d_1)$ is the remaining fraction of $E^Q(S_T)$ to pay for $E^Q\left(Y_T^1\right)$

Alternatively, notice that $X_T^1+Y_T^1=\begin{cases} S_T & \text{If } S_T<K \\ S_T & \text{If } S_T>K \end{cases}=S_T$

Hence $E^Q\left(X_T^1\right)+E^Q\left(Y_T^1\right)=E^Q(S_T)$. So $E^Q(S_T)$ consists of two parts, one to pay for $E^Q\left(X_T^1\right)=E^Q(S_T)\Phi(d_1)$ and the other to pay for $E^Q\left(Y_T^1\right)=E^Q(S_T)\Phi(-d_1)$, where $\Phi(d_1)$ and $\Phi(-d_1)$ represent the fraction of $E^Q(S_T)$ used to pay for $E^Q\left(X_T^1\right)$ and $E^Q\left(Y_T^1\right)$ respectively.

Problem 12.10.

A silly option gives its owner the right to receive $\ln S_T$ at T by paying K. The assumptions under the Black-Scholes formula hold. Calculate the option price.

Solution.

The option payoff at T is $X_T = \begin{cases} \ln S_T - K & \text{If } \ln S_T > K \\ 0 & \text{If } \ln S_T \leq K \end{cases}$. The option price at time zero is $V = e^{-rT} E^Q (X_T)$.

$S_T = S_0 \exp\left[\left(r - \delta - 0.5\sigma^2\right) T + \sigma\sqrt{T} z\right]$

Solve $\ln S_T > K$ $\quad$ $\ln S_T = \ln S_0 + \left[\left(r - \delta - 0.5\sigma^2\right) T + \sigma\sqrt{T} z\right] > K$

$\rightarrow \quad z > \dfrac{K - \ln S_0 - \left(r - \delta - 0.5\sigma^2\right) T}{\sigma\sqrt{T}} = -d_2^*$

where $d_2^* = \dfrac{\ln S_0 - K + \left(r - \delta - 0.5\sigma^2\right) T}{\sigma\sqrt{T}}$

$E^Q (X_T) = \int_{-d_2^*}^{\infty} (\ln S_T - K) f(z)\, dz = \int_{-d_2^*}^{\infty} \left[\ln S_0 + \left(r - \delta - 0.5\sigma^2\right) T + \sigma\sqrt{T} z - K\right] f(z)\, dz$

To calculate $\int_{-d_2^*}^{\infty} z f(z)\, dz$, notice that for $z \sim N(0, 1)$ and $f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-0.5z^2}$

$zf(z) = z\dfrac{1}{\sqrt{2\pi}} e^{-0.5z^2} = -\dfrac{d}{dz}\left(\dfrac{1}{\sqrt{2\pi}} e^{-0.5z^2}\right) = -\dfrac{d}{dz}[f(z)]$

Hence $\int_d^{\infty} zf(z)\, dz = \int_d^{\infty} -\dfrac{d}{dz}[f(z)]\, dz = \int_d^{\infty} -d[f(z)] = -f(z)|_d^{\infty} = f(d) - f(\infty) = f(d) - 0 = f(d)$

$$\int_d^{\infty} zf(z)\, dz = f(d) = \frac{1}{\sqrt{2\pi}} e^{-0.5d^2} \tag{12.11}$$

$\int_{-d_2^*}^{\infty} \left[\ln S_0 + \left(r - \delta - 0.5\sigma^2\right) T + \sigma\sqrt{T} z - K\right] f(z)\, dz$
$= \int_{-d_2^*}^{\infty} \left[\ln S_0 + \left(r - \delta - 0.5\sigma^2\right) T - K\right] f(z)\, dz + \sigma\sqrt{T} \int_{-d_2^*}^{\infty} zf(z)\, dz$
$= \left[\ln S_0 + \left(r - \delta - 0.5\sigma^2\right) T - K\right] \Phi(d_2^*) + \sigma\sqrt{T} f(-d_2^*)$

The option price is
$V = e^{-rT} E^Q (X_T) = e^{-rT} \left[\ln S_0 + \left(r - \delta - 0.5\sigma^2\right) T - K\right] \times \Phi(d_2^*) + \sigma\sqrt{T} f(-d_2^*)$

Problem 12.11.

A silly option gives its owner the right to receive $\sqrt{S_T}$ at T by paying K. The assumptions under the Black-Scholes formula hold. Calculate the option price.

Solution.

We'll calculate a generic option where the option owner has the right to get cash equal to $(S_T)^m = S_T^m$ (where $m \neq 0$) at T by paying K.

The option payoff at T is $X_T = \begin{cases} S_T^m - K & \text{If } S_T^m > K \\ 0 & \text{If } S_T^m \leq K \end{cases}$. The option price at time zero is $V = e^{-rT} E^Q(X_T)$.

$$S_T = S_0 \exp\left[\left(r - \delta - 0.5\sigma^2\right)T + \sigma\sqrt{T}z\right]$$

$$S_T^m = S_0^m \exp\left[m\left(r - \delta - 0.5\sigma^2\right)T + m\sigma\sqrt{T}z\right]$$

Solve $S_T^m > K$: $\qquad \ln S_T^m > \ln K$

$$\rightarrow \qquad z > \frac{\ln K - \ln S_0^m - m\left(r - \delta - 0.5\sigma^2\right)T}{m\sigma\sqrt{T}} = -d_2^*$$

$$d_2^* = -\frac{\ln K - \ln S_0^m - m\left(r - \delta - 0.5\sigma^2\right)T}{m\sigma\sqrt{T}} = \frac{\ln\dfrac{S_0^m}{K} + m\left(r - \delta - 0.5\sigma^2\right)T}{m\sigma\sqrt{T}}$$

$$E^Q(X_T) = \int_{-d_2^*}^{\infty}\left(S_T^m - K\right)f(z)\,dz = \int_{-d_2^*}^{\infty} S_T^m f(z)\,dz - K\int_{-d_2^*}^{\infty} f(z)\,dz$$

$$K\int_{-d_2^*}^{\infty} f(z)\,dz = K\Phi\left(d_2^*\right)$$

$$\int_{-d_2^*}^{\infty} S_T^m f(z)\,dz = \int_{-d_2^*}^{\infty} S_0^m \exp\left[m\left(r - \delta - 0.5\sigma^2\right)T + m\sigma\sqrt{T}z\right] f(z)\,dz$$
$$= S_0^m e^{m\left(r-\delta-0.5\sigma^2\right)T} e^{0.5m^2\sigma^2 T}\Phi\left(m\sigma\sqrt{T} + d_2^*\right)$$

Set $m\sigma\sqrt{T} + d_2^* = d_1^*$

$$V = e^{-rT}E^Q(X_T) = e^{-rT}S_0^m e^{m\left(r-\delta-0.5\sigma^2\right)T} e^{0.5m^2\sigma^2 T}\Phi\left(d_1^*\right) - e^{-rT}K\Phi\left(d_2^*\right)$$

If $m = 0.5$, then

$$V = e^{-rT}\sqrt{S_0}e^{0.5\left(r-\delta-0.5\sigma^2\right)T}e^{0.5^3\sigma^2 T}\Phi\left(d_1^*\right) - e^{-rT}K\Phi\left(d_2^*\right)$$

where $d_2^* = \dfrac{\ln\dfrac{\sqrt{S_0}}{K} + 0.5\left(r - \delta - 0.5\sigma^2\right)T}{0.5\sigma\sqrt{T}}$ $\qquad d_1^* = 0.5\sigma\sqrt{T} + d_2^*$

Problem 12.12.

A special contract pays $(S_T - K)^2$ at T if $S_T > K$. The assumptions under the Black-Scholes formula hold. Calculate the contract price.

Solution.

Method 1 Borrow as much as you can from the Black-Scholes formula

The contract payoff is $X_T = \begin{cases} (S_T - K)^2 = S_T^2 - 2KS_T + K^2 & \text{If } S_T > K \\ 0 & \text{If } S_T < K \end{cases}$

Consider three contracts:

- Contract #1 pays S_T^2 if $S_T > K$. The payoff is $Y_T^1 = \begin{cases} S_T^2 & \text{If } S_T > K \\ 0 & \text{If } S_T < K \end{cases}$.
- Contract #2 pays S_T if $S_T > K$. The payoff is $Y_T^2 = \begin{cases} S_T & \text{If } S_T > K \\ 0 & \text{If } S_T < K \end{cases}$
- Contract #3 pays K at T if $S_T > K$. The payoff is $Y_T^3 = \begin{cases} K & \text{If } S_T > K \\ 0 & \text{If } S_T < K \end{cases}$

Let V_1, V_2, and V_3 represent the price of the above contracts respectively. Let V represent the price of the contract with payoff X_T.

We replicate X_T by buying 1 unit of Contract #1, selling $2K$ units of Contract #2, and buying K units of Contract #3:
$X_T = Y_T^1 - 2KY_T^2 + KY_T^3 \qquad \Longrightarrow V = V_1 - 2KV_2 + KV_3$

V_2 is equal to the first component of the Black-Scholes call price:
$V_2 = S_0 e^{-\delta T}\Phi(d_1)$ where $d_2 = \sigma\sqrt{T} + d_2$

V_3 is equal to the second component of the Black-Scholes call price:

$$V_3 = Ke^{-rT}\Phi(d_2) \text{ where } d_2 = \frac{\ln \dfrac{S_0}{K} + \left(r - \delta - \dfrac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

The remaining work is to calculate $V_1 = e^{-rT}E^Q\left(Y_T^1\right) = e^{-rT}\int_{-d_2}^{\infty} S_T^2(z)\, f(z)\, dz$.

$\int_{-d_2}^{\infty} S_T^2(z)\, f(z)\, dz = \int_{-d_2}^{\infty}\left[S_0 e^{\left(r-\delta-0.5\sigma^2\right)T+\sigma\sqrt{T}z}\right]^2 f(z)\, dz$
$= S_0^2 e^{2\left(r-\delta-0.5\sigma^2\right)T+0.5\left(2\sigma\sqrt{T}\right)^2}\Phi\left(2\sigma\sqrt{T}+d_2\right)$
$= S_0^2 e^{2(r-\delta)T+\sigma^2 T}\Phi\left(2\sigma\sqrt{T}+d_2\right)$
$V = S_0^2 e^{(r-2\delta)T}e^{\sigma^2 T}\Phi\left(2\sigma\sqrt{T}+d_2\right) - 2KS_0e^{-\delta T}\Phi(d_1) + K^2e^{-rT}\Phi(d_2)$

By the way, from the previous problem, we know the price of an option that allows you to pay K and receive S_T^m is $e^{-rT}S_0^m e^{m\left(r-\delta-0.5\sigma^2\right)T}e^{0.5m^2\sigma^2 T}\Phi(d_1^*) -$

$e^{-rT}K\Phi(d_2^*)$. You may be attempted to use this formula to calculate V_1 by setting $m=2$, but that won't work. The contract in the previous problem pays S_T^2 if $S_T^2 > K$. In contrast, Contract #1 in this problem pays S_T^2 if $S_T > K$ (so $S_T^2 > K$ vs. $S_T > K$).

Method 2 Calculate from scratch

$S_T > K$ is the same as $z > -d_2$. The contract payoff is as $X_T(z) = \begin{cases} S_T^2(z) - 2KS_T(z) + K^2 & \text{If } z > -d_2 \\ 0 & \text{If } z < -d_2 \end{cases}$

The contract price is $V = e^{-rT}E^Q(X_T)$

$$V = e^{-rT}E^Q(X_T)$$

$E^Q(X_T) = \int_{-\infty}^{\infty} X_T(z) f(z)\,dz = \int_{-d_2}^{\infty} S_T^2(z) f(z)\,dz - 2K\int_{-d_2}^{\infty} S_T(z) f(z)\,dz + K^2\int_{-d_2}^{\infty} f(z)\,dz$

Evaluating this integral, you should get

$$V = S_0^2 e^{(r-2\delta)T} e^{\sigma^2 T}\Phi\left(2\sigma\sqrt{T} + d_2\right) - 2KS_0e^{-\delta T}\Phi(d_1) + K^2e^{-rT}\Phi(d_2)$$

Problem 12.13.

A special contract pays, at T, the greater of S_T and K. Calculate its price.

Solution.

The payoff is

$$X_T(z) = \max(S_T, K) = \begin{cases} S_T(z) & \text{If } S_T > K \\ K & \text{If } S_T < K \end{cases} = \begin{cases} S_T(z) & \text{If } z > -d_2 \\ K & \text{If } z < -d_2 \end{cases}$$

Method 1

$$E^Q(X_T) = \int_{-d_2}^{\infty} S_T(z) f(z)\,dz + \int_{-\infty}^{-d_2} Kf(z)\,dz = S_0e^{(r-\delta)T}N(d_1) + KN(-d_2)$$

$$V = e^{-rT}E^Q(X_T) = S_0e^{-\delta T}N(d_1) + Ke^{-rT}N(-d_2)$$

Method 2

$$X_T(z) = \max(S_T, K) = \max(S_T - K, 0) + K$$

$\max(S_T - K, 0)$ is the payoff of a call option. It's price is $S_0e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$.

The price of K is Ke^{-rT}. Hence the contract price is

$$V = S_0e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT}$$
$$= S_0e^{-\delta T}N(d_1) + Ke^{-rT}[1 - N(d_2)] = S_0e^{-\delta T}N(d_1) + Ke^{-rT}N(-d_2)$$

Problem 12.14.

A special contract pays $|S_T - K|$ at T. Calculate its price.

Solution.

The payoff is

$$X_T(z) = |S_T - K| = \begin{cases} S_T - K & \text{If } S_T > K \\ K - S_T & \text{If } S_T < K \end{cases}$$

Method 1

Notice that $S_T - K$ If $S_T > K$ is the payoff of a call option; $K - S_T$ If $S_T < K$ is the payoff of a put option. Hence the contract price is

$$V = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) + K e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$$
$$= S_0 e^{-\delta T} [N(d_1) - N(-d_1)] - K e^{-rT} [N(d_2) - N(-d_2)]$$
$$= S_0 e^{-\delta T} [2N(d_1) - 1] - K e^{-rT} [2N(d_2) - 1]$$

Method 2

$$|S_T - K| = 2\max(S_T - K, 0) - (S_T - K)$$

$2\max(S_T - K, 0)$ is twice the payoff of a call option. Its price is $2\left[S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)\right]$
The price of $(S_T - K)$ is $S_0 e^{-\delta T} - K e^{-rT}$

The contract price is

$$V = 2\left[S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)\right] - \left(S_0 e^{-\delta T} - K e^{-rT}\right)$$

Problem 12.15.

Derive the gap call price formula DM 14.15.

Solution.

K_1 is the payment amount; K_2 is the payment trigger. The payoff is:

$$X_T(z) = \begin{cases} S_T(z) - K_1 & \text{If } S_T > K_2 \\ 0 & \text{If } S_T < K_2 \end{cases} = \begin{cases} S_T(z) - K_1 & \text{If } z > -d_2 \\ 0 & \text{If } z < -d_2 \end{cases}$$

where $d_2 = \dfrac{\ln \dfrac{S_0}{K_2} + \left(r - \delta - \dfrac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}$ is calculated by solving:

$$S_T = S_0 \exp\left[\left(r - \delta - 0.5\sigma^2\right) T + \sigma\sqrt{T} z\right] > K_2$$

$$z > \frac{\ln \dfrac{K_2}{S_0} - \left(r - \delta - \dfrac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} = -\frac{\ln \dfrac{S_0}{K_2} + \left(r - \delta - \dfrac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} = -d_2$$

$$E^Q(X_T) = \int_{-d_2}^{\infty} [S_T(z) - K_1] f(z)\, dz = \int_{-d_2}^{\infty} S_T(z) f(z)\, dz - \int_{-d_2}^{\infty} K_1 f(z)\, dz$$
$$= S_0 e^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2) \quad \text{where } d_1 = d_2 + \sigma\sqrt{T}$$