

Deeper Understanding, Faster Calc: SOA MFE and CAS Exam 3F

Yufeng Guo

Contents

Introduction	ix
9 Parity and other option relationships	1
9.1 Put-call parity	1
9.1.1 Option on stocks	1
9.1.2 Options on currencies	11
9.1.3 Options on bonds	13
9.1.4 Generalized parity and exchange options	13
9.1.5 Comparing options with respect to style, maturity, and strike	19
10 Binomial option pricing: I	35
10.1 One-period binomial model: simple examples	35
10.2 General one-period binomial model	36
10.2.1 Two or more binomial trees	49
10.2.2 Options on stock index	64
10.2.3 Options on currency	67
10.2.4 Options on futures contracts	71
11 Binomial option pricing: II	79
11.1 Understanding early exercise	79
11.2 Understanding risk-neutral probability	80
11.2.1 Pricing an option using real probabilities	81
11.2.2 Binomial tree and lognormality	88
11.2.3 Estimate stock volatility	91
11.3 Stocks paying discrete dividends	95
11.3.1 Problems with discrete dividend tree	97
11.3.2 Binomial tree using prepaid forward	98
12 Black-Scholes	105
12.1 Introduction to the Black-Scholes formula	105
12.1.1 Call and put option price	105
12.1.2 When is the Black-Scholes formula valid?	107
12.2 Derive the Black-Scholes formula	107

12.3	Applying the formula to other assets	121
12.3.1	Black-Scholes formula in terms of prepaid forward price	121
12.3.2	Options on stocks with discrete dividends	121
12.3.3	Options on currencies	122
12.3.4	Options on futures	123
12.4	Option the Greeks	124
12.4.1	Delta	124
12.4.2	Gamma	125
12.4.3	Vega	125
12.4.4	Theta	126
12.4.5	Rho	126
12.4.6	Psi	126
12.4.7	Greek measures for a portfolio	126
12.4.8	Option elasticity and volatility	127
12.4.9	Option risk premium and Sharpe ratio	128
12.4.10	Elasticity and risk premium of a portfolio	129
12.5	Profit diagrams before maturity	129
12.5.1	Holding period profit	129
12.5.2	Calendar spread	132
12.6	Implied volatility	133
12.6.1	Calculate the implied volatility	133
12.6.2	Volatility skew	134
12.6.3	Using implied volatility	135
12.7	Perpetual American options	135
12.7.1	Perpetual calls and puts	135
12.7.2	Barrier present values	140
13	Market-making and delta-hedging	143
13.1	Delta hedging	143
13.2	Examples of Delta hedging	143
13.3	Textbook Table 13.2	152
13.4	Textbook Table 13.3	154
13.5	Mathematics of Delta hedging	155
13.5.1	Delta-Gamma-Theta approximation	155
13.5.2	Understanding the market maker's profit	156
14	Exotic options: I	159
14.1	Asian option (i.e. average options)	159
14.1.1	Characteristics	159
14.1.2	Examples	160
14.1.3	Geometric average	160
14.1.4	Payoff at maturity T	160
14.2	Barrier option	161
14.2.1	Knock-in option	161
14.2.2	Knock-out option	161

14.2.3	Rebate option	161
14.2.4	Barrier parity	162
14.2.5	Examples	162
14.3	Compound option	163
14.4	Gap option	165
14.4.1	Definition	165
14.4.2	Pricing formula	165
14.4.3	How to memorize the pricing formula	165
14.5	Exchange option	166
18	Lognormal distribution	169
18.1	Normal distribution	169
18.2	Lognormal distribution	170
18.3	Lognormal model of stock prices	170
18.4	Lognormal probability calculation	171
18.4.1	Lognormal confidence interval	172
18.4.2	Conditional expected prices	176
18.4.3	Black-Scholes formula	177
18.5	Estimating the parameters of a lognormal distribution	177
18.6	How are asset prices distributed	179
18.6.1	Histogram	179
18.6.2	Normal probability plots	180
18.7	Sample problems	182
19	Monte Carlo valuation	187
19.1	Example 1 Estimate $E(e^z)$	187
19.2	Example 2 Estimate pi	191
19.3	Example 3 Estimate the price of European call or put options	194
19.4	Example 4 Arithmetic and geometric options	198
19.5	Efficient Monte Carlo valuation	207
19.5.1	Control variance method	207
19.6	Antithetic variate method	211
19.7	Stratified sampling	213
19.7.1	Importance sampling	213
19.8	Sample problems	213
20	Brownian motion and Ito's Lemma	219
20.1	Introduction	219
20.1.1	Big picture	220
20.2	Brownian motion	221
20.2.1	Deterministic process vs. stochastic process	221
20.2.2	Definition of Brownian motion	222
20.2.3	Martingale	224
20.2.4	Properties of Brownian motion	231

20.2.5	Arithmetic Brownian motion and Geometric Brownian motion	236
20.2.6	Ornstein-Uhlenbeck process	237
20.3	Definition of the stochastic calculus	238
20.3.1	Why stochastic calculus	238
20.4	Properties of the stochastic calculus	248
20.5	Ito's lemma	251
20.5.1	Multiplication rules	251
20.5.2	Ito's lemma	252
20.6	Geometric Brownian motion revisited	253
20.6.1	Relative importance of drift and noise term	254
20.6.2	Correlated Ito processes	254
20.7	Three worlds: R , Q , and V	261
20.8	Sharpe ratio	305
20.9	Girsanov's theorem	307
20.10	Risk neutral process	317
20.11	Valuing a claim on S^a	329
20.11.1	Process followed by S^a	329
20.11.2	Formula for $S^A(t)$ and $E[S^A(t)]$	330
20.11.3	Expected return of a claim on $S^A(t)$	331
20.11.4	Specific examples	331
21	Black-Scholes equation	341
21.1	Differential equations and valuation under certainty	341
21.1.1	Valuation equation	341
21.1.2	Bonds	342
21.1.3	Dividend paying stock	342
21.2	Black-Scholes equation	342
21.2.1	How to derive Black-Scholes equation	342
21.2.2	Verifying the formula for a derivative	343
21.2.3	Black-Scholes equation and equilibrium returns	346
21.3	Risk-neutral pricing	348
22	Exotic options: II	349
22.1	All-or-nothing options	349
23	Volatility	351
24	Interest rate models	353
24.1	Market-making and bond pricing	353
24.1.1	Review of duration and convexity	353
24.1.2	Interest rate is not so simple	360
24.1.3	Impossible bond pricing model	361
24.1.4	Equilibrium equation for bonds	365
24.1.5	Delta-Gamma approximation for bonds	370
24.2	Equilibrium short-rate bond price models	371

24.2.1	Arithmetic Brownian motion (i.e. Merton model)	371
24.2.2	Rendleman-Bartter model	372
24.2.3	Vasicek model	373
24.2.4	CIR model	395
24.3	Bond options, caps, and the Black model	400
24.3.1	Black formula	400
24.3.2	Interest rate caplet	403
24.4	Binomial interest rate model	404
24.5	Black-Derman-Toy model	408

Introduction

This study guide is for SOA MFE and CAS Exam 3F. Before you start, make sure you have the following items:

1. Derivatives Markets, the 2nd edition.
2. Errata of Derivatives Markets. You can download the errata at http://www.kellogg.northwestern.edu/faculty/mcdonald/htm/typos2e_01.html. Don't miss the errata about the textbook pages 780 through 788.
3. Download the syllabus from the SOA or CAS website.
4. Download the sample MFE problems and solutions from the SOA website.
5. Download the recent SOA MFE and CAS Exam 3 problems.

Please report any errors to yufeng.guo.actuary@gmail.com.

Chapter 12

Black-Scholes

You probably have memorized the famous Black-Scholes call and put price formulas and can readily calculate the price of a plain vanilla European call or put option. But what if SOA throws a tricky derivative at you? Here are a few examples of ad hoc contracts:

- An option allows you to pay K and receive $\ln S_T$ at T . What's its price?
- An option allows you to pay K and receive $\sqrt{S_T}$ at T . What's its price?
- An option pays $(S_t - K)^2$ at T only if $S_T > K$. What's its price?

To tackle non-standard derivatives, you need to do more than memorize the Black-Scholes formula. In this chapter, you'll learn how to derive the Black-Scholes formula from the ground up and how to price an ad hoc contract.

The math behind the Black-Scholes formula is simple. All you need to know is (1) some Exam P level calculus, and (2) the risk neutral pricing (an option is worth its expected payoff discounted at the risk free rate).

First, though, let's review the basics of the Black-Scholes formula.

12.1 Introduction to the Black-Scholes formula

12.1.1 Call and put option price

The price of a European call option is:

$$C(S, K, \sigma, r, T, \delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \quad (12.1)$$

The price of a European put option is:

$$P(S, K, \sigma, r, T, \delta) = -Se^{-\delta T} N(-d_1) + Ke^{-rT} N(-d_2) \quad (12.2)$$

$$d_1 = \frac{\ln \frac{S}{K} + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad (12.3)$$

$$d_2 = d_1 - \sigma\sqrt{T} \quad (12.4)$$

Notations used in Equation 12.1, 12.3, and 12.4:

- S , the current stock price (i.e. the stock price when the option is written)
- K , the strike price
- r , the continuously compounded risk-free interest rate per year
- δ , the continuously compounded dividend rate per year
- σ , the annualized standard deviation of the continuously compounded stock return (i.e. stock volatility)
- T , option expiration time
- $N(d) = P(z \leq d)$ where z is a standard normal random variable
- $C(S, K, \sigma, r, T, \delta)$, the price of a European call option with parameters $(S, K, \sigma, r, T, \delta)$
- $P(S, K, \sigma, r, T, \delta)$, the price of a European put option with parameters $(S, K, \sigma, r, T, \delta)$

Tip 12.1.1. To help memorize Equation 12.2, we can rewrite Equation 12.2 similar to Equation 12.1 as $P(S, K, \sigma, r, T, \delta) = (-S)e^{-\delta T}N(-d_1) - (-K)e^{-rT}N(-d_2)$. In other words, change S , K , d_1 , and d_2 in Equation 12.1 and you'll get Equation 12.2.

Example 12.1.1. Reproduce the textbook example 12.1. This is the recap of the information. $S = 41$, $K = 40$, $r = 0.08$, $\sigma = 0.3$, $T = 0.25$ (i.e. 3 months), and $\delta = 0$. Calculate the price of the price of a European call option.

$$\begin{aligned} d_1 &= \frac{\ln \frac{S}{K} + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \\ &= \frac{\ln \frac{41}{40} + \left(0.08 - 0 + \frac{1}{2} \times 0.3^2\right)0.25}{0.3\sqrt{0.25}} = 0.3730 \\ d_2 &= d_1 - \sigma\sqrt{T} = 0.3730 - 0.3\sqrt{0.25} = 0.2230 \\ N(d_1) &= 0.6454 \quad N(d_2) = 0.5882 \\ C &= 41e^{-0(0.25)}0.6454 - 40e^{-0.08(0.25)}0.5882 = 3.399 \end{aligned}$$

Example 12.1.2. Reproduce the textbook example 12.2. This is the recap of the information. $S = 41$, $K = 40$, $r = 0.08$, $\sigma = 0.3$, $T = 0.25$ (i.e. 3 months), and $\delta = 0$. Calculate the price of the price of a European put option.

$$\begin{aligned} N(-d_1) &= 1 - N(d_1) = 1 - 0.6454 = 0.3546 \\ N(-d_2) &= 1 - N(d_2) = 1 - 0.5882 = 0.4118 \\ P &= -41e^{-0(0.25)}0.3546 + 40e^{-0.08(0.25)}0.4118 = 1.607 \end{aligned}$$

12.1.2 When is the Black-Scholes formula valid?

Assumptions under the Black-Scholes formula:

Assumptions about the distribution of stock price:

1. Continuously compounded returns on the stock are normally distributed (i.e. stock price is lognormally distributed) and independent over time
2. The volatility of the continuously compounded returns is known and constant
3. Future dividends are known, either as a dollar amount (i.e. D and T_D are known in advance) or as a fixed dividend yield (i.e. δ is a known constant)

Assumptions about the economic environment

1. The risk-free rate is known and fixed (i.e. r is a known constant)
2. There are no transaction costs or taxes
3. It's possible to short-sell costlessly and to borrow at the risk-free rate

12.2 Derive the Black-Scholes formula

By learning how to derive the Black-Scholes formula, we'll be able to remove the black-box behind the formula and recreate the formula instantly. First, some basics.

What's the density function of a standard normal random variable?

$$\text{If } z \sim N(0, 1), \text{ then } f(z) = \frac{1}{\sqrt{2\pi}}e^{-0.5z^2}, \Phi(a) = P(z \leq a) = \int_{-\infty}^a f(z) dz$$

How can we convert a normal variable to a standard normal variable?

$$\text{If } Z \sim N(\mu, \sigma^2), \text{ then set } z = \frac{Z - \mu}{\sigma}; Z = \mu + z\sigma$$

What's the stock price S_t under the Black-Scholes assumption in the real world?

$$\text{DM 20.13: } S_t = S_0 \exp[(\alpha - \delta - 0.5\sigma^2)t + \sigma\sqrt{t}z], z \sim N(0, 1)$$

What's S_t under the Black-Scholes assumption in the risk neutral world Q ?

$$\text{Set } \alpha = r. \text{ Then } S_t \stackrel{Q}{=} S_0 \exp[(r - \delta - 0.5\sigma^2)t + \sigma\sqrt{t}z], z \sim N(0, 1)$$

Next, let's derive some normal random variable related integral shortcuts. The first shortcut. For a standard normal random variable $z \sim N(0, 1)$ and a constant d ,

$$\int_d^\infty f(z) dz = P(z > d) = \Phi(-d) \quad (12.5)$$

Proof. Clearly, $\int_d^\infty f(z) dz = P(z > d)$. We just need to prove that $P(z > d) = \Phi(-d)$.

$$P(z > d) = 1 - P(z < d) = 1 - \Phi(d).$$

Notice that $P(z = d) = 0$ (the probability for a continuous random variable to take on a single value is zero).

Hence $P(z > d) = 1 - P(z < d) - P(z = d) = 1 - P(z < d)$. Then using the formula $\Phi(d) + \Phi(-d) = 1$, we get 12.5.

The second integral shortcut:

$$\int e^{\sigma z} f(z) dz = e^{-0.5\sigma^2} \int \frac{1}{\sqrt{2\pi}} e^{-0.5(z-\sigma)^2} dz \quad (12.6)$$

Proof. $z \sim N(0, 1)$ and $f(z) = \frac{1}{\sqrt{2\pi}} e^{-0.5z^2}$

$$\begin{aligned} \int e^{\sigma z} f(z) dz &= \int e^{\sigma z} \frac{1}{\sqrt{2\pi}} e^{-0.5z^2} dz = \int \frac{1}{\sqrt{2\pi}} e^{-0.5z^2 + \sigma z} dz \\ -0.5z^2 + \sigma z &= -0.5(z^2 - 2\sigma z + \sigma^2) + 0.5\sigma^2 = -0.5(z - \sigma)^2 + 0.5\sigma^2 \end{aligned}$$

$$\int \frac{1}{\sqrt{2\pi}} e^{-0.5z^2 + \sigma z} dz = \int \frac{1}{\sqrt{2\pi}} e^{-0.5(z-\sigma)^2 + 0.5\sigma^2} dz = e^{0.5\sigma^2} \int \frac{1}{\sqrt{2\pi}} e^{-0.5(z-\sigma)^2} dz$$

The third shortcut is:

$$\int_d^\infty e^{\sigma z} f(z) dz = e^{0.5\sigma^2} \Phi(\sigma - d) \quad (12.7)$$

Proof. $\int_d^\infty e^{\sigma z} f(z) dz = e^{-0.5\sigma^2} \int_d^\infty \frac{1}{\sqrt{2\pi}} e^{-0.5(z-\sigma)^2} dz$

$$\text{Set } t = z - \sigma. \quad \int_d^\infty \frac{1}{\sqrt{2\pi}} e^{-0.5(z-\sigma)^2} dz = \int_{d-\sigma}^\infty \frac{1}{\sqrt{2\pi}} e^{-0.5t^2} dt$$

$\frac{1}{\sqrt{2\pi}} e^{-0.5t^2}$ is the density of $t \sim N(0, 1)$

$$\rightarrow \int_{d-\sigma}^\infty \frac{1}{\sqrt{2\pi}} e^{-0.5t^2} dt = \Phi(-(d-\sigma)) = \Phi(\sigma - d).$$

$$\rightarrow \int_d^\infty e^{\sigma z} f(z) dz = e^{0.5\sigma^2} \Phi(\sigma - d).$$

Problem 12.1.

For a normal random variable $X \sim N(\mu, \sigma^2)$, calculate $E(e^X)$.

Solution.

$$X = \mu + \sigma z, \quad z \sim N(0, 1).$$

$$E(e^X) = \int_{-\infty}^{\infty} e^{\mu + \sigma z} f(z) dz = e^{\mu} \int_{-\infty}^{\infty} e^{\sigma z} f(z) dz$$

Use Equation $\int_d^{\infty} e^{\sigma z} f(z) dz = e^{0.5\sigma^2} \Phi(\sigma - d)$, set $d = -\infty$:

$$\int_{-\infty}^{\infty} e^{\sigma z} f(z) dz = e^{0.5\sigma^2} \Phi(\sigma - (-\infty)) = e^{0.5\sigma^2} \Phi(\infty) = e^{0.5\sigma^2} \times 1 = e^{0.5\sigma^2}$$

$$\implies E(e^X) = e^{\mu} e^{0.5\sigma^2} = e^{\mu + 0.5\sigma^2} = e^{E(X) + 0.5\text{Var}(X)}$$

$$E\left(e^{X \sim N(\mu, \sigma^2)}\right) = e^{E(X) + 0.5\text{Var}(X)} = e^{\mu + 0.5\sigma^2} \quad (12.8)$$

Problem 12.2.

For a normal random variable $X \sim N(\mu, \sigma^2)$, calculate $\int_d^{\infty} e^X f(z) dz$ where $z \sim N(0, 1)$ and $f(z) = \frac{1}{\sqrt{2\pi}} e^{-0.5z^2}$.

Solution.

$$\int_d^{\infty} e^{x(z)} f(z) dz = \int_d^{\infty} e^{\mu + \sigma z} f(z) dz = e^{\mu} \int_d^{\infty} e^{\sigma z} f(z) dz = e^{\mu} e^{0.5\sigma^2} \Phi(\sigma - d) = e^{\mu + 0.5\sigma^2} \Phi(\sigma - d) = E(e^X) \Phi(\sigma - d)$$

$$\int_d^{\infty} e^X f(z) dz = E(e^X) \Phi(\text{std dev of } X - d) = E(e^X) \Phi(\sigma - d) \quad (12.9)$$

Tip 12.2.1. Whenever you need to calculate $\int_d^{\infty} e^X f(z) dz$ where $X \sim N(\mu, \sigma^2)$, first calculate $E(e^X) = e^{\mu + 0.5\sigma^2}$. Next, multiply $E(e^X)$ by $\Phi(\sigma - d)$.

Problem 12.3.

For a normal random variable $X \sim N(\mu, \sigma^2)$, calculate $\int_{-\infty}^d e^X f(z) dz$ where $z \sim N(0, 1)$ and $f(z) = \frac{1}{\sqrt{2\pi}} e^{-0.5z^2}$.

Solution.

$$\int_{-\infty}^d e^X f(z) dz = \int_{-\infty}^{\infty} e^X f(z) dz - \int_d^{\infty} e^X f(z) dz = E(e^X) - E(e^X) \Phi(\sigma - d) = E(e^X) [1 - \Phi(\sigma - d)] = E(e^X) \Phi[-(\sigma - d)]$$

$$\int_{-\infty}^d e^X f(z) dz = E(e^X) \Phi[-(\text{std dev of } X - d)] = E(e^X) \Phi[-(\sigma - d)] \quad (12.10)$$

Tip 12.2.2. Whenever you need to calculate $\int_{-\infty}^d e^X f(z) dz$ where $X \sim N(\mu, \sigma^2)$, first calculate $E(e^X) = e^{\mu + 0.5\sigma^2}$. Next, multiply $E(e^X)$ by $\Phi[-(\sigma - d)]$.

Problem 12.4.

Derive the Black-Scholes call option formula 12.1.

Solution.

Consider two contracts:

- Contract #1 pays S_T at T if $S_T > K$. The payoff at T is $X_T^1 = \begin{cases} S_T & \text{If } S_T > K \\ 0 & \text{If } S_T \leq K \end{cases}$.
- Contract #2 pays K at T if $S_T > K$. The payoff at T is $X_T^2 = \begin{cases} K & \text{If } S_T > K \\ 0 & \text{If } S_T \leq K \end{cases}$.

Let V_1 and V_2 represent the price of the Contract #1 and #2 respectively. Under the risk neutral pricing, the price of a contract is just the expected payoff discounted at the risk free rate. Hence $V_1 = e^{-rT} E^Q(X_T^1)$ and $V_2 = e^{-rT} E^Q(X_T^2)$, where Q is the risk-neutral world.

Since $X_T^1 - X_T^2 = \begin{cases} S_T - K & \text{If } S_T > K \\ 0 & \text{If } S_T \leq K \end{cases}$ is the payoff of a call option, the call option price is equal to $V = V_1 - V_2$.

$$S_T \stackrel{Q}{=} S_0 e^X, \quad X \stackrel{Q}{\sim} N[\mu = (r - \delta - 0.5\sigma^2)T, \text{Var} = \sigma^2 T]$$

$$\text{Solve } S_T(z) = S_0 \exp\left[(r - \delta - 0.5\sigma^2)T + \sigma\sqrt{T}z\right] > K$$

$$\rightarrow z > \frac{\ln \frac{K}{S_0} - (r - \delta - 0.5\sigma^2)T}{\sigma\sqrt{T}} = -d_2$$

$$\text{Notice } d_2 = \frac{\ln \frac{S_0}{K} + (r - \delta - 0.5\sigma^2)T}{\sigma\sqrt{T}} \text{ in the Black-Scholes formula. Then}$$

$S_T(z) > K$ is the same as $z > -d_2$. Hence we can write X_T^1 and X_T^2 as follows:

$$X_T^1 = \begin{cases} S_T & \text{If } S_T > K \\ 0 & \text{If } S_T \leq K \end{cases} = \begin{cases} S_T & \text{If } z > -d_2 \\ 0 & \text{If } z \leq -d_2 \end{cases}$$

$$X_T^2 = \begin{cases} K & \text{If } S_T > K \\ 0 & \text{If } S_T \leq K \end{cases} = \begin{cases} K & \text{If } z > -d_2 \\ 0 & \text{If } z \leq -d_2 \end{cases}$$

$$\begin{aligned} E^Q(X_T^2) &= \int_{-\infty}^{-d_2} 0 f(z) dz + \int_{-d_2}^{\infty} K f(z) dz = \int_{-d_2}^{\infty} K f(z) dz = K \int_{-d_2}^{\infty} f(z) dz = \\ &K \Phi(d_2) \\ \Rightarrow V_2 &= e^{-rT} E^Q(Y) = K e^{-rT} \Phi(d_2) \end{aligned}$$

Notice $\Phi(d_2) = \int_{-d_2}^{\infty} f(z) dz = P^Q(S_T > K)$ is the risk neutral probability of $S_T > K$.

$$\begin{aligned} E^Q(X_T^1) &= \int_{S_T > K} S_T(z) f(z) dz + \int_{S_T < K} 0 f(z) dz = \int_{S_T > K} S_T(z) f(z) dz = \\ &\int_{-d_2}^{\infty} S_T(z) f(z) dz = S_0 \int_{-d_2}^{\infty} e^{X(z)} f(z) dz = S_0 \int_{-d_2}^{\infty} e^{(r-\delta-0.5\sigma^2)T + \sigma\sqrt{T}z} f(z) dz \end{aligned}$$

Use the formula: $\int_d^{\infty} e^X f(z) dz = E(e^X) \Phi(\text{std dev of } X - d)$

$$\Rightarrow S_0 \int_{-d_2}^{\infty} e^X f(z) dz = S_0 E(e^X) \Phi(\text{std dev of } X + d_2)$$

$$E(e^X) = e^{E(X) + 0.5 \text{Var}(X)} = e^{(r-\delta-0.5\sigma^2)T + 0.5\sigma^2 T} = e^{(r-\delta)T}$$

$$\Phi(\text{std dev of } X + d_2) = \Phi(\sigma\sqrt{T} + d_2) = \Phi(d_1)$$

$$\Rightarrow E^Q(X_T^1) = S_0 e^{(r-\delta)T} \Phi(d_1)$$

$$\Rightarrow V = V_1 - V_2 = S_0 e^{-\delta T} \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

Problem 12.5.

Derive the Black-Scholes put option formula 12.2.

Solution.

Consider two contracts:

- Contract #1 pays S_T at T if $S_T < K$. The payoff at T is $Y_T^1 = \begin{cases} S_T & \text{If } S_T < K \\ 0 & \text{If } S_T \geq K \end{cases}$.
- Contract #2 pays K at T if $S_T < K$. The payoff at T is $Y_T^2 = \begin{cases} K & \text{If } S_T < K \\ 0 & \text{If } S_T \geq K \end{cases}$.

Let V_1 and V_2 represent the price of the Contract #1 and #2 respectively. $V_1 = e^{-rT} E^Q(Y_T^1)$ and $V_2 = e^{-rT} E^Q(Y_T^2)$, where Q is the risk-neutral world. Since $Y_T^2 - Y_T^1 = \begin{cases} K - S_T & \text{If } S_T < K \\ 0 & \text{If } S_T \geq K \end{cases}$ is the payoff of a put option, the put option price is equal to $V = V_2 - V_1$.

Solve $S_T(z) > K$.

$$\rightarrow z < \frac{\ln \frac{K}{S_0} - (r - \delta - 0.5\sigma^2)T}{\sigma\sqrt{T}} = -d_2$$

Then $S_T(z) < K$ is the same as $z < -d_2$. Hence we write X_T^1 and X_T^2 as follows:

$$Y_T^1 = \begin{cases} S_T & \text{If } S_T < K \\ 0 & \text{If } S_T \geq K \end{cases} = \begin{cases} S_T & \text{If } z < -d_2 \\ 0 & \text{If } z \geq -d_2 \end{cases}$$

$$Y_T^2 = \begin{cases} K & \text{If } S_T < K \\ 0 & \text{If } S_T \geq K \end{cases} = \begin{cases} K & \text{If } z < -d_2 \\ 0 & \text{If } z \geq -d_2 \end{cases}$$

$$\begin{aligned} E^Q(Y_T^2) &= \int_{-\infty}^{-d_2} K f(z) dz + \int_{-d_2}^{\infty} 0 f(z) dz = \int_{-\infty}^{-d_2} K f(z) dz = K\Phi(-d_2) \\ \Rightarrow V_2 &= e^{-rT} E^Q(Y) = Ke^{-rT}\Phi(-d_2) \end{aligned}$$

$\Phi(-d_2) = \int_{-\infty}^{-d_2} f(z) dz = P^Q(S_T < K)$ is the risk neutral probability of $S_T < K$.

$$\begin{aligned} E^Q(Y_T^1) &= \int_{-\infty}^{-d_2} S_T(z) f(z) dz + \int_{-d_2}^{\infty} 0 f(z) dz = \int_{-\infty}^{-d_2} S_T(z) f(z) dz = \\ &= \int_{-\infty}^{-d_2} S_0 e^X f(z) dz = S_0 \int_{-\infty}^{-d_2} e^X f(z) dz \end{aligned}$$

$$\text{where } X \stackrel{Q}{\sim} N[\mu = (r - \delta - 0.5\sigma^2)T, \text{Var} = \sigma^2 T]$$

$$\begin{aligned} \int_{-\infty}^{-d_2} e^X f(z) dz &= E(e^X) \Phi[-(\text{std dev of } X - (-d_2))] = E(e^X) \Phi\left[-\left(\sigma\sqrt{T} + d_2\right)\right] = \\ &= E(e^X) \Phi(-d_1) \\ E(e^X) &= e^{E(X) + 0.5\text{Var}(X)} = e^{(r - \delta - 0.5\sigma^2)T + 0.5\sigma^2 T} = e^{(r - \delta)T} \\ \Rightarrow E^Q(Y_T^1) &= e^{(r - \delta)T} \Phi(-d_1) \quad V_2 = e^{-rT} [E^Q(Y_T^1)] = e^{-\delta T} \Phi(-d_1) \\ \Rightarrow V &= V_2 - V_1 = e^{-rT} K \Phi(-d_2) - S_0 e^{-\delta T} \Phi(-d_1) \end{aligned}$$

Problem 12.6.

What's the meaning of $\Phi(d_2)$ and $\Phi(-d_2)$ in the Black-Scholes formula?

Solution.

$\Phi(d_2) = P^Q(S_T > K)$ is the risk neutral probability of $S_T > K$.
 $\Phi(-d_2) = P^Q(S_T < K)$ is the risk neutral probability of $S_T < K$.

Problem 12.7.

Since $\Phi(d_2)$ is the risk neutral probability of $S_T > K$, I thought the call price should be $S_0 e^{-\delta T} \Phi(d_2) - K e^{-rT} \Phi(d_2)$, but the Black-Scholes formula is $S_0 e^{-\delta T} \Phi(d_1) - K e^{-rT} \Phi(d_2)$. Why?

Solution.

The wrong formula $C = S_0 e^{-\delta T} \Phi(d_2) - K e^{-rT} \Phi(d_2) = (S_0 e^{-\delta T} - K e^{-rT}) \Phi(d_2)$ can easily produce a negative or zero call price when $S_0 \leq K$.

For example, set $\delta = r = 0$ and $S_0 = K$. The wrong formula is $C = (S_0 - K) \Phi(d_2) = 0$, but a call price is always positive.

Here's another example. To simplify calculation, set $\delta = 0$, $r = 0.06$, $S_0 = 50$, $K = 100$, $\sigma = 1$, $T = 1$.

$$d_2 = \frac{\ln \frac{S}{K} + \left(r - \delta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\ln \frac{50}{100} + (0.06 - 0 - 0.5 \times 1^2)1}{1\sqrt{1}} = -1.13315$$

$$d_1 = d_2 + \sigma\sqrt{T} = -1.13315 + 1\sqrt{1} = -0.13315$$

$$N(d_2) = \text{NormalDist}(-1.13315) = 0.12858$$

$$N(d_1) = \text{NormalDist}(-0.13315) = 0.44704$$

Notice that $N(d_1 = d_2 + \sigma\sqrt{T}) > N(d_2)$ because the cumulative density function $N(z)$ is an increasing function of z .

The correct call price is

$$C = S_0 e^{-\delta T} \Phi(d_1) - K e^{-rT} \Phi(d_2) = 50 \times 0.44704 - 100e^{-0.06} \times 0.12858 = 10.24$$

The wrong call price is

$$C = S_0 e^{-\delta T} \Phi(d_2) - K e^{-rT} \Phi(d_2) = (50 - 100e^{-0.06}) 0.12858 = -5.68$$

Third example. Set $\delta = 0$, $r = 0.06$, $S_0 = 1$, $K = 100$, $\sigma = 1$, $T = 1$

$$d_2 = \frac{\ln \frac{S}{K} + \left(r - \delta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\ln \frac{1}{100} + (0.06 - 0 - 0.5 \times 1^2)1}{1\sqrt{1}} = -5.04517$$

$$d_1 = d_2 + \sigma\sqrt{T} = -5.04517 + 1\sqrt{1} = -4.04517$$

$$N(d_2) = \text{NormalDist}(-5.04517) = 2.26559 \times 10^{-7}$$

$$N(d_1) = \text{NormalDist}(-4.04517) = 2.61426 \times 10^{-5}$$

$$\frac{N(d_1)}{N(d_2)} = \frac{2.61426 \times 10^{-5}}{2.26559 \times 10^{-7}} = 115.3898$$

The correct call price is

$$C = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2) = \Phi(d_2) \left(S_0 \frac{\Phi(d_1)}{\Phi(d_2)} - K e^{-rT} \right) = 2.26559 \times 10^{-7} (1 \times 115.3898 - 100e^{-0.06}) = 4.8 \times 10^{-6}$$

In this example, as K becomes much greater than S_0 , $\frac{\Phi(d_1)}{\Phi(d_2)}$ gets big as well, making the call price $\Phi(d_2) \left(S_0 \frac{\Phi(d_1)}{\Phi(d_2)} - K e^{-rT} \right)$ positive.

In contrast, the wrong formula $C = (S_0 - K e^{-rT}) \Phi(d_2) = (1 - 100e^{-0.06}) 2.26559 \times 10^{-7} = -2.11 \times 10^{-5}$ produces a negative call price.

Similarly, though $\Phi(-d_2) = P^Q(S_T < K)$ is the risk neutral probability of $S_T < K$, the put price is **NOT** $V = K e^{-rT} \Phi(-d_2) - S_0 e^{-\delta T} N(-d_2)$ but $V = K e^{-rT} \Phi(-d_2) - S_0 e^{-\delta T} N(-d_1)$. Since $-d_2 > -d_2 = -d_1 - \sigma\sqrt{T}$, we have $\Phi(-d_2) > N(-d_1)$. This makes the put price $K e^{-rT} \Phi(-d_2) - S_0 e^{-\delta T} N(-d_1)$ positive.

Problem 12.8.

Verify that

- the first term of the Black-Scholes call price formula $S_0 e^{-\delta T} N(d_1)$ is equal to $e^{-rT} [E^Q(S_T | S_T > K) E^Q(S_T > K)]$, not equal to $e^{-rT} [E^Q(S_T) E^Q(S_T > K)]$
- the 2nd term of the Black-Scholes put price formula $S_0 e^{-\delta T} N(-d_1)$ is equal to $e^{-rT} [E^Q(S_T | S_T < K) E^Q(S_T < K)]$, not equal to $e^{-rT} [E^Q(S_T) E^Q(S_T < K)]$

Solution.

$S_T > K$ is the same as $z > -d_2$

Notice that

$$\underbrace{\int_{-\infty}^{\infty} S_T(z) f(z) dz}_{E^Q(S_T)} = \underbrace{\int_{-d_2}^{\infty} S_T(z) f(z) dz}_{E^Q(X_T^1) = E^Q(S_T | S_T > K) P^Q(S_T > K)} + \underbrace{\int_{-\infty}^{-d_2} S_T(z) f(z) dz}_{E^Q(Y_T^1) = E^Q(S_T | S_T < K) P^Q(S_T < K)}$$

where X_T^1 and Y_T^1 are the payoffs of the following contracts:

- Contract #1 pays S_T at T if $S_T > K$. The payoff at T is $X_T^1 = \begin{cases} S_T & \text{If } S_T > K \\ 0 & \text{If } S_T < K \end{cases}$.
- Contract #2 pays S_T at T if $S_T < K$. The payoff at T is $Y_T^1 = \begin{cases} S_T & \text{If } S_T < K \\ 0 & \text{If } S_T > K \end{cases}$

$$\begin{aligned}
E^Q(X_T^1) &= E^Q(S_T | S_T > K) E^Q(S_T > K) = S_0 E^Q(e^X) N(d_1) = S_0 e^{(r-\delta)T} N(d_1) \\
\implies e^{-rT} [E^Q(X_T^1)] &= e^{-rT} [E^Q(S_T | S_T > K) E^Q(S_T > K)] = S_0 e^{-\delta T} N(d_1) \\
E^Q(Y_T^1) &= E^Q(S_T | S_T < K) E^Q(S_T < K) = S_0 E^Q(e^X) N(-d_1) = S_0 e^{(r-\delta)T} N(-d_1) \\
\implies e^{-rT} [E^Q(Y_T^1)] &= e^{-rT} [E^Q(S_T | S_T < K) E^Q(S_T < K)] = S_0 e^{-\delta T} N(-d_1)
\end{aligned}$$

Problem 12.9.

What's the meaning of $N(d_1)$ and $N(-d_1)$?

Solution.

Later in the chapter about the V world (where $S_t e^{\delta t}$ is used as the numeraire) we'll learn that $N(d_1) = P^V(S_T > K)$ (the V world probability of $S_T > K$) and $N(-d_1) = P^V(S_T < K)$ (the V world probability of $S_T < K$). Now let's interpret $N(d_1)$ and $N(-d_1)$ in a different way.

Consider two contracts:

- Contract #1 pays S_T at T if $S_T > K$. The payoff at T is $X_T^1 = \begin{cases} S_T & \text{If } S_T > K \\ 0 & \text{If } S_T < K \end{cases}$.
- Contract #2 pays S_T at T if $S_T < K$. The payoff at T is $Y_T^1 = \begin{cases} S_T & \text{If } S_T < K \\ 0 & \text{If } S_T > K \end{cases}$.

Notice that

$$E^Q(S_T) = \int_{-\infty}^{\infty} S_T(z) f(z) dz = \underbrace{\int_{-d_2}^{\infty} S_T(z) f(z) dz}_{E^Q(X_T^1) = E^Q(S_T) \Phi(d_1)} + \underbrace{\int_{-\infty}^{-d_2} S_T(z) f(z) dz}_{E^Q(Y_T^1) = E^Q(S_T) \Phi(-d_1)}$$

$E^Q(S_T)$ consists of two parts, one to pay for $E^Q(X_T^1) = E^Q(S_T) \Phi(d_1)$ and the other to pay for $E^Q(Y_T^1) = E^Q(S_T) \Phi(-d_1)$. We see that

- $\Phi(d_1)$ is the fraction of $E^Q(S_T)$ to pay for $E^Q(X_T^1)$, and
- $\Phi(-d_1) = 1 - \Phi(d_1)$ is the remaining fraction of $E^Q(S_T)$ to pay for $E^Q(Y_T^1)$.

Alternatively, notice that $X_T^1 + Y_T^1 = \begin{cases} S_T & \text{If } S_T < K \\ S_T & \text{If } S_T > K \end{cases} = S_T$

Hence $E^Q(X_T^1) + E^Q(Y_T^1) = E^Q(S_T)$. So $E^Q(S_T)$ consists of two parts, one to pay for $E^Q(X_T^1) = E^Q(S_T) \Phi(d_1)$ and the other to pay for $E^Q(Y_T^1) = E^Q(S_T) \Phi(-d_1)$, where $\Phi(d_1)$ and $\Phi(-d_1)$ represent the fraction of $E^Q(S_T)$ used to pay for $E^Q(X_T^1)$ and $E^Q(Y_T^1)$ respectively.

Problem 12.10.

A silly option gives its owner the right to receive $\ln S_T$ at T by paying K . The assumptions under the Black-Scholes formula hold. Calculate the option price.

Solution.

The option payoff at T is $X_T = \begin{cases} \ln S_T - K & \text{If } \ln S_T > K \\ 0 & \text{If } \ln S_T \leq K \end{cases}$. The option price at time zero is $V = e^{-rT} E^Q (X_T)$.

$$S_T = S_0 \exp \left[(r - \delta - 0.5\sigma^2) T + \sigma\sqrt{T}z \right]$$

$$\text{Solve } \ln S_T > K \quad \ln S_T = \ln S_0 + \left[(r - \delta - 0.5\sigma^2) T + \sigma\sqrt{T}z \right] > K$$

$$\rightarrow \quad z > \frac{K - \ln S_0 - (r - \delta - 0.5\sigma^2) T}{\sigma\sqrt{T}} = -d_2^*$$

$$\text{where } d_2^* = \frac{\ln S_0 - K + (r - \delta - 0.5\sigma^2) T}{\sigma\sqrt{T}}$$

$$E^Q (X_T) = \int_{-d_2^*}^{\infty} (\ln S_T - K) f(z) dz = \int_{-d_2^*}^{\infty} \left[\ln S_0 + (r - \delta - 0.5\sigma^2) T + \sigma\sqrt{T}z - K \right] f(z) dz$$

To calculate $\int_{-d_2^*}^{\infty} z f(z) dz$, notice that for $z \sim N(0, 1)$ and $f(z) = \frac{1}{\sqrt{2\pi}} e^{-0.5z^2}$

$$zf(z) = z \frac{1}{\sqrt{2\pi}} e^{-0.5z^2} = -\frac{d}{dz} \left(\frac{1}{\sqrt{2\pi}} e^{-0.5z^2} \right) = -\frac{d}{dz} [f(z)]$$

$$\text{Hence } \int_d^{\infty} zf(z) dz = \int_d^{\infty} -\frac{d}{dz} [f(z)] dz = \int_d^{\infty} -d[f(z)] = -f(z)|_d^{\infty} = f(d) - f(\infty) = f(d) - 0 = f(d)$$

$$\int_d^{\infty} zf(z) dz = f(d) = \frac{1}{\sqrt{2\pi}} e^{-0.5d^2} \quad (12.11)$$

$$\begin{aligned} & \int_{-d_2^*}^{\infty} \left[\ln S_0 + (r - \delta - 0.5\sigma^2) T + \sigma\sqrt{T}z - K \right] f(z) dz \\ &= \int_{-d_2^*}^{\infty} [\ln S_0 + (r - \delta - 0.5\sigma^2) T - K] f(z) dz + \sigma\sqrt{T} \int_{-d_2^*}^{\infty} zf(z) dz \\ &= [\ln S_0 + (r - \delta - 0.5\sigma^2) T - K] \Phi(d_2^*) + \sigma\sqrt{T} f(-d_2^*) \end{aligned}$$

The option price is

$$V = e^{-rT} E^Q (X_T) = e^{-rT} [\ln S_0 + (r - \delta - 0.5\sigma^2) T - K] \times \Phi(d_2^*) + \sigma\sqrt{T} f(-d_2^*)$$

Problem 12.11.

A silly option gives its owner the right to receive $\sqrt{S_T}$ at T by paying K . The assumptions under the Black-Scholes formula hold. Calculate the option price.

Solution.

We'll calculate a generic option where the option owner has the right to get cash equal to $(S_T)^m = S_T^m$ (where $m \neq 0$) at T by paying K .

The option payoff at T is $X_T = \begin{cases} S_T^m - K & \text{If } S_T^m > K \\ 0 & \text{If } S_T^m \leq K \end{cases}$. The option price at time zero is $V = e^{-rT} E^Q (X_T)$.

$$S_T = S_0 \exp \left[(r - \delta - 0.5\sigma^2) T + \sigma\sqrt{T}z \right]$$

$$S_T^m = S_0^m \exp \left[m(r - \delta - 0.5\sigma^2) T + m\sigma\sqrt{T}z \right]$$

Solve $S_T^m > K$: $\ln S_T^m > \ln K$

$$\rightarrow z > \frac{\ln K - \ln S_0^m - m(r - \delta - 0.5\sigma^2) T}{m\sigma\sqrt{T}} = -d_2^*$$

$$d_2^* = -\frac{\ln K - \ln S_0^m - m(r - \delta - 0.5\sigma^2) T}{m\sigma\sqrt{T}} = \frac{\ln \frac{S_0^m}{K} + m(r - \delta - 0.5\sigma^2) T}{m\sigma\sqrt{T}}$$

$$E^Q (X_T) = \int_{-d_2^*}^{\infty} (S_T^m - K) f(z) dz = \int_{-d_2^*}^{\infty} S_T^m f(z) dz - K \int_{-d_2^*}^{\infty} f(z) dz$$

$$K \int_{-d_2^*}^{\infty} f(z) dz = K\Phi(d_2^*)$$

$$\begin{aligned} \int_{-d_2^*}^{\infty} S_T^m f(z) dz &= \int_{-d_2^*}^{\infty} S_0^m \exp \left[m(r - \delta - 0.5\sigma^2) T + m\sigma\sqrt{T}z \right] f(z) dz \\ &= S_0^m e^{m(r - \delta - 0.5\sigma^2)T} e^{0.5m^2\sigma^2T} \Phi \left(m\sigma\sqrt{T} + d_2^* \right) \end{aligned}$$

$$\text{Set } m\sigma\sqrt{T} + d_2^* = d_1^*$$

$$V = e^{-rT} E^Q (X_T) = e^{-rT} S_0^m e^{m(r - \delta - 0.5\sigma^2)T} e^{0.5m^2\sigma^2T} \Phi(d_1^*) - e^{-rT} K\Phi(d_2^*)$$

If $m = 0.5$, then

$$V = e^{-rT} \sqrt{S_0} e^{0.5(r - \delta - 0.5\sigma^2)T} e^{0.5^3\sigma^2T} \Phi(d_1^*) - e^{-rT} K\Phi(d_2^*)$$

$$\text{where } d_2^* = \frac{\ln \frac{\sqrt{S_0}}{K} + 0.5(r - \delta - 0.5\sigma^2) T}{0.5\sigma\sqrt{T}} \quad d_1^* = 0.5\sigma\sqrt{T} + d_2^*$$

Problem 12.12.

A special contract pays $(S_T - K)^2$ at T if $S_T > K$. The assumptions under the Black-Scholes formula hold. Calculate the contract price.

Solution.

Method 1 **Borrow as much as you can from the Black-Scholes formula**

$$\text{The contract payoff is } X_T = \begin{cases} (S_T - K)^2 = S_T^2 - 2KS_T + K^2 & \text{If } S_T > K \\ 0 & \text{If } S_T < K \end{cases}$$

Consider three contracts:

- Contract #1 pays S_T^2 if $S_T > K$. The payoff is $Y_T^1 = \begin{cases} S_T^2 & \text{If } S_T > K \\ 0 & \text{If } S_T < K \end{cases}$.
- Contract #2 pays S_T if $S_T > K$. The payoff is $Y_T^2 = \begin{cases} S_T & \text{If } S_T > K \\ 0 & \text{If } S_T < K \end{cases}$.
- Contract #3 pays K at T if $S_T > K$. The payoff is $Y_T^3 = \begin{cases} K & \text{If } S_T > K \\ 0 & \text{If } S_T < K \end{cases}$.

Let V_1, V_2 , and V_3 represent the price of the above contracts respectively. Let V represent the price of the contract with payoff X_T .

We replicate X_T by buying 1 unit of Contract #1, selling $2K$ units of Contract #2, and buying K units of Contract #3:

$$X_T = Y_T^1 - 2KY_T^2 + KY_T^3 \implies V = V_1 - 2KV_2 + KV_3$$

V_2 is equal to the first component of the Black-Scholes call price:

$$V_2 = S_0 e^{-\delta T} \Phi(d_1) \text{ where } d_2 = \sigma\sqrt{T} + d_1$$

V_3 is equal to the second component of the Black-Scholes call price:

$$V_3 = K e^{-rT} \Phi(d_2) \text{ where } d_2 = \frac{\ln \frac{S_0}{K} + \left(r - \delta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

The remaining work is to calculate $V_1 = e^{-rT} E^Q(Y_T^1) = e^{-rT} \int_{-d_2}^{\infty} S_T^2(z) f(z) dz$.

$$\int_{-d_2}^{\infty} S_T^2(z) f(z) dz = \int_{-d_2}^{\infty} \left[S_0 e^{(r-\delta-0.5\sigma^2)T + \sigma\sqrt{T}z} \right]^2 f(z) dz$$

$$= S_0^2 e^{2(r-\delta-0.5\sigma^2)T + 0.5(2\sigma\sqrt{T})^2} \Phi(2\sigma\sqrt{T} + d_2)$$

$$= S_0^2 e^{2(r-\delta)T + \sigma^2 T} \Phi(2\sigma\sqrt{T} + d_2)$$

$$V = S_0^2 e^{2(r-\delta)T} e^{\sigma^2 T} \Phi(2\sigma\sqrt{T} + d_2) - 2KS_0 e^{-\delta T} \Phi(d_1) + K^2 e^{-rT} \Phi(d_2)$$

By the way, from the previous problem, we know the price of an option that allows you to pay K and receive S_T^m is $e^{-rT} S_0^m e^{m(r-\delta-0.5\sigma^2)T} e^{0.5m^2\sigma^2 T} \Phi(d_1^*) -$

$e^{-rT}K\Phi(d_2^*)$. You may be attempted to use this formula to calculate V_1 by setting $m = 2$, but that won't work. The contract in the previous problem pays S_T^2 if $S_T^2 > K$. In contrast, Contract #1 in this problem pays S_T^2 if $S_T > K$ (so $S_T^2 > K$ vs. $S_T > K$).

Method 2 Calculate from scratch

$S_T > K$ is the same as $z > -d_2$. The contract payoff is as $X_T(z) =$

$$\begin{cases} S_T^2(z) - 2KS_T(z) + K^2 & \text{If } z > -d_2 \\ 0 & \text{If } z < -d_2 \end{cases}$$

 The contract price is $V = e^{-rT}E^Q(X_T)$

$$V = e^{-rT}E^Q(X_T)$$

$$E^Q(X_T) = \int_{-\infty}^{\infty} X_T(z) f(z) dz = \int_{-d_2}^{\infty} S_T^2(z) f(z) dz - 2K \int_{-d_2}^{\infty} S_T(z) f(z) dz + K^2 \int_{-d_2}^{\infty} f(z) dz$$

Evaluating this integral, you should get

$$V = S_0^2 e^{(r-2\delta)T} e^{\sigma^2 T} \Phi(2\sigma\sqrt{T} + d_2) - 2KS_0 e^{-\delta T} \Phi(d_1) + K^2 e^{-rT} \Phi(d_2)$$

Problem 12.13.

A special contract pays, at T , the greater of S_T and K . Calculate its price.

Solution.

The payoff is

$$X_T(z) = \max(S_T, K) = \begin{cases} S_T(z) & \text{If } S_T > K \\ K & \text{If } S_T < K \end{cases} = \begin{cases} S_T(z) & \text{If } z > -d_2 \\ K & \text{If } z < -d_2 \end{cases}$$

Method 1

$$E^Q(X_T) = \int_{-d_2}^{\infty} S_T(z) f(z) dz + \int_{-\infty}^{-d_2} K f(z) dz = S_0 e^{(r-\delta)T} N(d_1) + K N(-d_2)$$

$$V = e^{-rT}E^Q(X_T) = S_0 e^{-\delta T} N(d_1) + K e^{-rT} N(-d_2)$$

Method 2

$$X_T(z) = \max(S_T, K) = \max(S_T - K, 0) + K$$

$\max(S_T - K, 0)$ is the payoff of a call option. Its price is $S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)$.

The price of K is $K e^{-rT}$. Hence the contract price is

$$\begin{aligned} V &= S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) + K e^{-rT} \\ &= S_0 e^{-\delta T} N(d_1) + K e^{-rT} [1 - N(d_2)] = S_0 e^{-\delta T} N(d_1) + K e^{-rT} N(-d_2) \end{aligned}$$

Problem 12.14.

A special contract pays $|S_T - K|$ at T . Calculate its price.

Solution.

The payoff is

$$X_T(z) = |S_T - K| = \begin{cases} S_T - K & \text{If } S_T > K \\ K - S_T & \text{If } S_T < K \end{cases}$$

Method 1

Notice that $S_T - K$ If $S_T > K$ is the payoff of a call option; $K - S_T$ If $S_T < K$ is the payoff of a put option. Hence the contract price is

$$\begin{aligned} V &= S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) + K e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1) \\ &= S_0 e^{-\delta T} [N(d_1) - N(-d_1)] - K e^{-rT} [N(d_2) - N(-d_2)] \\ &= S_0 e^{-\delta T} [2N(d_1) - 1] - K e^{-rT} [2N(d_2) - 1] \end{aligned}$$

Method 2

$$|S_T - K| = 2 \max(S_T - K, 0) - (S_T - K)$$

$2 \max(S_T - K, 0)$ is twice the payoff of a call option. Its price is $2 [S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)]$
The price of $(S_T - K)$ is $S_0 e^{-\delta T} - K e^{-rT}$

The contract price is

$$V = 2 [S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)] - (S_0 e^{-\delta T} - K e^{-rT})$$

Problem 12.15.

Derive the gap call price formula DM 14.15.

Solution.

K_1 is the payment amount; K_2 is the payment trigger. The payoff is:

$$X_T(z) = \begin{cases} S_T(z) - K_1 & \text{If } S_T > K_2 \\ 0 & \text{If } S_T < K_2 \end{cases} = \begin{cases} S_T(z) - K_1 & \text{If } z > -d_2 \\ 0 & \text{If } z < -d_2 \end{cases}$$

$$\text{where } d_2 = \frac{\ln \frac{S_0}{K_2} + \left(r - \delta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \text{ is calculated by solving:}$$

$$S_T = S_0 \exp \left[\left(r - \delta - 0.5\sigma^2\right)T + \sigma\sqrt{T}z \right] > K_2$$

$$z > \frac{\ln \frac{K_2}{S_0} - \left(r - \delta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = -\frac{\ln \frac{S_0}{K_2} + \left(r - \delta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = -d_2$$

$$\begin{aligned} E^Q(X_T) &= \int_{-d_2}^{\infty} [S_T(z) - K_1] f(z) dz = \int_{-d_2}^{\infty} S_T(z) f(z) dz - \int_{-d_2}^{\infty} K_1 f(z) dz \\ &= S_0 e^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2) \quad \text{where } d_1 = d_2 + \sigma\sqrt{T} \end{aligned}$$