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1st Edition

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Contents

Lesson 2

Project Analysis

Reading: *Corporate Finance* 8.5, IFM 21-18 2–3

This lesson begins the corporate finance part of the course. From here up to and including Lesson 13, when not told otherwise, assume that interest rates are annual effective.

2.1 NPV

When a company considers embarking on a project, it must verify that this project will meet the company's financial goals. The measure we will use for this is *NPV*, or net present value. To compute the NPV we calculate the *free cash flows* of the project. The free cash flows are the cash amounts generated by the project itself, both positive and negative, year by year. Cash flows do not include non-cash accounting items, such as depreciation.[1] The free cash flows also do not include cash flows from financing used to support the project. If a loan is taken to pay the project's initial expenses, neither the loan nor interest on the loan is part of free cash flows. Free cash flows are purely cash generated by the project itself.

The NPV is the present value, at the start of the project, of the project's free cash flows. At what interest rate is the NPV calculated? Usually the NPV is calculated at the interest rate the company must pay to finance the project. In other words, the NPV is calculated at the interest rate that has to be paid to investors in order to get them to invest in the project. This interest rate is called the *cost of capital*. We will discuss what the cost of capital should be in the following lessons.

Example 2A A life insurance company is considering developing a new Universal Life product. It will cost \$5 million, payable immediately, to develop this product, and developing the product will take a year. The company estimates free cash flows will be \$−1 million at the end of the first year, followed by \$1 million per year for 5 years, and will then decrease at a compounded rate of 10% per year after that.

The company's cost of capital is 12%.

Calculate the NPV of the project.

Answer: The NPV generated during the first 6 years is

$$-5{,}0000{,}000-\frac{1{,}000{,}000}{1.12}+1{,}000{,}000\left(\frac{a_{\overline{5}|0.12}}{1.12}\right)=-5{,}797{,}194+1{,}000{,}000\left(\frac{1-1/1.12^5}{0.12(1.12)}\right)=-2{,}674{,}307$$

After 6 years, free cash flows form a geometric series with first term $900{,}000/1.12^7$ and ratio $0.9/1.12$. The NPV generated after year 6, in millions, is

$$9000{,}000\left(\frac{1/1.12^7}{1-0.9/1.12}\right)=2{,}072{,}582$$

Total NPV is $-2{,}674{,}307+2{,}027{,}582=\boxed{\mathbf{-601{,}725}}$. □

Companies should invest in a project only if NPV > 0. Otherwise they destroy the value of the company.

[1] However, for insurance products, they include changes in reserves. A company must set aside cash to support the reserves, although this cash may be invested.

If we assume that free cash flows are constant, they form a perpetuity. As you learned in Financial Mathematics, the present value at of an immediate perpetuity of 1 per year is $1/i$, where i is the interest rate. If the free cash flows are 1 in the first year and grow at compounded rate g, then their present value is

$$\begin{aligned}\text{NPV} &= \sum_{n=1}^{\infty}\frac{(1+g)^{n-1}}{(1+i)^n}\\ &= \frac{1/(1+i)}{1-(1+g)/(1+i)}\\ &= \frac{1/(1+i)}{(i-g)/(1+i)} = \frac{1}{i-g}\end{aligned} \tag{2.1}$$

Quiz 2-1 A company is considering a project. This project will require an investment of 10 million immediately and will generate free cash flows of 1 million per year at the end of one year, increasing at a compounded rate of 3% per year perpetually.

The cost of capital is 9%.

Calculate the NPV of the project.

2.2 Project analysis

2.2.1 Break-even analysis

Companies analyze the risk in a project. One way to analyze the risk is to vary the assumptions used to calculate the NPV with the changed assumptions. *Break-even analysis* consists of determining the value of each assumption parameter for which the NPV is 0, assuming that the other assumption parameters are at their baseline values.

Calculation of *IRR* is an example of break-even analysis. IRR, the internal rate of return, is an alternative profit measure to NPV. The IRR is the interest rate r such that the present value of free cash flows at r is 0. Assuming the usual pattern of negative free cash flows initially followed by positive free cash flows, IRR is the highest interest rate for which the NPV is at least 0. Thus IRR is the highest interest rate for which the company breaks even.

A similar analysis can be done for the other parameters. A break-even analysis calculates the break-even level of number of sales, expenses, sales price, level of cash flows per year, and any other parameter.

Example 2B A project requires an immediate investment of 19 million. It is expected to generate free cash flows of 2 million per year at the end of the first year, growing 2% per year perpetually. The cost of capital is 12%.

Perform a break-even analysis on the rate of growth of free cash flows.

Answer: Let g be the growth rate. We want to solve $-15 + \frac{2}{0.12-g} = 0$

$$\begin{aligned}-19 + \frac{2}{0.12-g} &= 0\\ \frac{2}{0.12-g} &= 19\\ 0.12 - g &= \frac{2}{19}\\ g &= \boxed{\mathbf{0.01474}}\end{aligned}$$

□

Quiz 2-2 A project to develop a new product requires an immediate investment of 9 million. It will then generate free cash flows of 1 million per year starting with the end of the first year, until the product becomes obsolete and cannot be sold. The cost of capital is 10%.

Perform a break-even analysis on the number of years the product must sell.

2.2.2 Sensitivity analysis

Sensitivity analysis consists of calculating the change in the NPV resulting from a change in a parameter. Typically one sets the parameter to its value in the worst possible case and the best possible case, and calculates the NPV for both cases. This analysis shows which parameters have the greatest impact on the NPV.

Example 2C A project to develop a new product requires an immediate investment of 15 million. Free cash flows generated by this project are 20% of sales. Sales are expected to be level, and to continue for a certain number of years, at which point the product becomes obsolete. The best and worst cases for each assumption are:

	Worst case	Baseline	Best case
Annual sales ($ million)	4	5	6
Number of years	3	5	7
Cost of capital	0.16	0.12	0.08

Perform a sensitivity analysis on the three factors listed in the table. Which factor is the NPV most sensitive to?

Answer: We'll do all calculations in millions.

For annual sales s:

$$\text{NPV} = -15 + s a_{\overline{5}|0.12} = -15 + s\left(\frac{1 - 1/1.12^5}{0.12}\right)$$

which is $-\$0.581$ million for $s = 4$ and \$6.629 million for $s = 6$, a variation of \$7.210 million

For number of years n:

$$\text{NPV} = -15 + 5a_{\overline{n}|0.12} = -15 + 5\left(\frac{1 - 1/1.12^n}{0.12}\right)$$

which is $-\$2.991$ million for $n = 3$ and 7.818 million for $n = 7$, a variation of \$10.809 million.

For cost of capital r:

$$\text{NPV} = -15 + 5a_{\overline{5}|r} = -15 + 5\left(\frac{1 - 1/(1+r)^5}{r}\right)$$

which is \$1.371 million for $r = 0.16$ and \$4.964 for $r = 0.08$, a variation of \$3.593 million.

We see that number of years of sales is the assumption to which NPV is most sensitive. □

2.2.3 Scenario analysis

Often parameters are correlated and should not be analyzed separately. For example, increasing the price of a product may lower sales. Scenario analysis consists of calculating the NPV for various scenarios. A scenario may vary two parameters in a consistent manner, leaving the other parameters unchanged if they are uncorrelated.

2.3 Risk measures

In the previous section we analyzed risk by varying parameters. An alternative method for analyzing risk is to assign a number to the project indicating its riskiness. This section discusses such risk measures. Each of these risk measures is a function from a random variable to a real number. The random variable may be profits, returns on investment, or aggregate loss amounts paid by an insurance company. Notice that the direction of risk for aggregate loss amounts is the opposite of profits or returns: the risk is that profits or returns are low and that loss amounts are high.

2.3.1 Four risk measures

We will discuss four risk measures: variance, semi-variance, VaR, and TVaR.

Variance

Variance is a popular risk measure and will be used in mean-variance portfolio theory, which we discuss starting in Lesson 5. If R is the random variable for the return on an investment, the mean return is μ and the variance is

$$\mathrm{Var}(R) = \sigma^2 = \mathbf{E}[(R-\mu)^2] = \mathbf{E}[R^2] - \mu^2 \tag{2.2}$$

An equivalent risk measure is the square root of the variance, or the standard deviation σ. We'll also use the notation $\mathrm{SD}(R)$ for the standard deviation.

$$\mathrm{SD}(R) = \sqrt{\mathrm{Var}(R)} = \sigma$$

The standard deviation of the rate of return is also called the *volatility* of the rate of return.

The variance may be estimated from a sample using the formula

$$\hat{\sigma}^2 = \sum_{i=1}^{n} \frac{(R_i - \bar{R})^2}{n}$$

where $\bar{R}$ is the sample mean. That is the formula given in the study note, but usually the denominator is $n-1$ instead of n to make this estimate unbiased. In fact, the formula for estimating volatility given in the Berk/DeMarzo textbook is equation (5.1) on page 59, and that formula divides by $n-1$.

Semi-variance

Since we are more concerned with underperformance than overperformance, at least for profits and rates of return on investments, we may prefer the downside semi-variance, which we'll refer to as the *semi-variance* for short, as a measure of risk. The semi-variance considers the square difference from the mean only when that difference is negative. It is defined by

$$\sigma_{SV}^2 = \mathbf{E}\Big[\min\big(0, (R-\mu)\big)^2\Big] \tag{2.3}$$

The semi-variance is positive even though it is based on negative differences from the mean, since the differences are squared. The square root of the downside semi-variance is the downside standard deviation.

Example 2D The random variable X has an exponential distribution with mean 1:

$$f_X(x) = e^{-x}, \qquad x > 0$$

Calculate the semi-variance of X.

Answer: We integrate $\min(0, x-1)^2$ over the density function. This minimum is 0 for $x > 1$, so we only integrate up to 1. We'll integrate by parts twice.

$$\begin{aligned}\int_0^1 (x-1)^2 e^{-x}\,\mathrm{d}x &= -(x-1)^2 e^{-x}\Big|_0^1 + 2\int_0^1 (x-1)e^{-x}\,\mathrm{d}x\\ &= 1 + 2\left(-(x-1)e^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,\mathrm{d}x\right)\\ &= 1 - 2 + 2(1-e^{-1}) = 1 - 2e^{-1} = \boxed{\mathbf{0.264241}}\end{aligned}$$

□

The sum of the downside semi-variance and the upside semi-variance is the variance:

$$\begin{aligned}\mathbf{E}[(X-\mu)^2] &= \mathbf{E}\left[\left(\min(0, X-\mu) + \max(0, X-\mu)\right)^2\right]\\ &= \mathbf{E}[\min(0, X-\mu)^2] + \mathbf{E}[\max(0, X-\mu)^2] + 2\,\mathbf{E}[\min(0, X-\mu)\max(0, X-\mu)]\end{aligned}$$

The first term of the last expression is the downside semi-variance. The second term is the upside semi-variance. The third term is 0, since 0 is either the minimum or the maximum, so one factor is always 0. However, the semi-variance doesn't have many of the nice properties of variance. For example, there is no easy formula for the semi-variance of a sum of two random variables.

The semi-variance may be estimated from a sample by

$$\hat{\sigma}^2_{SV} = \frac{1}{n}\sum_{i=1}^{n} \min\left(0, (R_i - \bar{R})\right)^2 \tag{2.4}$$

Quiz 2-3 You are given the following sample:

5 10 15 20 25

Calculate the sample semi-variance.

Value-at-Risk (VaR)

The VaR of a random variable X at level α is the 100α percentile of the random variable. For a continuous random variable, it is x such that $\Pr(X \le x) = \alpha$. For profits or rates of return, where the risk is that X is low, α is picked low, with values like 0.05, 0.025, 0.01, 0.005. For aggregate insurance losses, where the risk is that X is high, α is picked high, with values like 0.95, 0.975, 0.99, 0.995.

The VaR is calculated by inverting the cumulative distribution function:

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha) \tag{2.5}$$

Example 2E Profits have a distribution with the following density function:

$$f(x) = \frac{3}{(1+x)^4} \qquad x > 0$$

Calculate VaR of profits at the 0.01 level.

Answer: Integrate $f(x)$ to obtain the cumulative distribution function, then invert that function at 0.01.

$$\begin{aligned} F(x) &= \int_0^x \frac{3\,\mathrm{d}t}{(1+t)^4} = 1 - \frac{1}{(1+x)^3} \\ 1 - \frac{1}{(1+x)^3} &= 0.01 \\ 1 + x &= \sqrt[3]{1/0.99} = 1.003356 \\ x &= \boxed{\mathbf{0.003356}} \end{aligned}$$

□

To estimate VaR from a sample, the sample is ordered from lowest to highest, and then the 100α percentile is selected. This percentile is not well-defined since a sample is a discrete distribution, so some rule for selecting the percentile is needed. For example, if the sample is size 1000 and $\alpha = 0.05$, then one might set the sample VaR equal to the 50^{th} order statistic (most conservative), the 51^{st} order statistic, or some weighted average of the two, such as the smoothed empirical percentile defined in the Exam STAM syllabus.

Tail value-at-risk

While value-at-risk identifies the amount which returns or profits will exceed a great proportion ($1 - \alpha$ to be exact) of the time, it doesn't consider the severity of the downside risk in the remaining α of the time. Tail value-at-risk, also known as Conditional Tail Expectation (CTE) or Expected Shortfall measures this risk. It is defined as the expected value of the random variable given that it is below the 100α percentile for downside risk

$$\mathrm{TVaR}_\alpha(X) = \mathbf{E}[X \mid X < \mathrm{VaR}_\alpha(X)] = \frac{\int_{-\infty}^{\mathrm{VaR}_\alpha(X)} x f(x)\,\mathrm{d}x}{\alpha} \tag{2.6}$$

or the expected value of the random variable given that it is above the 100α percentile for upside risk, like aggregate losses

$$\mathrm{TVaR}_\alpha(X) = \mathbf{E}[X \mid X > \mathrm{VaR}_\alpha(X)] = \frac{\int_{\mathrm{Var}_\alpha(X)}^{\infty} x f(x)\,\mathrm{d}x}{1 - \alpha} \tag{2.7}$$

You should be able to figure out whether upside or downside risk is present based on what is being analyzed, but if not, if $\alpha < 0.5$, presumably the risk is downside and if $\alpha > 0.5$, presumably the risk is upside.

It may be difficult or impossible to evaluate the intergral needed to calculate TVaR.

TVaR can be estimated from a sample. Select the bottom or top α proportion of the items of the sample and calculate their mean. For example, if the sample is size 1000 and $\alpha = 0.05$, average the lowest 50 items of the sample to calculate downside risk.

2.3.2 Coherent risk measures

Let's list four desirable properties of a risk measure $g(X)$.

1. **Translation invariance.** Adding a constant to the random variable should add the same constant to the risk measure. Or:

   $$g(X + c) = g(X) + c$$

   This is reasonable, since a constant gain or loss generates no risk beyond its amount.

2. **Positive homogeneity.** Multiplying the random variable by a positive constant should multiply the risk measure by the same constant:

   $$g(cX) = cg(X)$$

   This is reasonable, since expressing the random variable in a different currency (for example) should not affect the risk measure.

3. **Subadditivity.** For any two random losses X and Y, the risk measure for $X+Y$ should not be greater than the sum of the risk measures for X and Y separately:

$$g(X+Y) \leq g(X) + g(Y)$$

   This is reasonable, since combining losses may result in diversification and reducing the total risk measure, but it should not be possible by breaking a risk into two sub-risks to reduce the total risk measure.

   This is for measuring upside risk. For measuring downside risk, the subadditivity property becomes $g(X+Y) \geq g(X) + g(Y)$.[2]

4. **Monotonicity.** For any two random losses X and Y, if X is always less than Y, or even if the probability that X is less than or equal to Y is 1, then the risk measure for X should be no greater than the risk measure for Y.

$$g(X) \leq g(Y) \text{ if } \Pr(X \leq Y) = 1$$

   This is reasonable, since X clearly has no more risk than Y.

   This is for measuring upside risk. For measuring downside risk, the monotonicity property becomes $g(X) \geq g(Y)$ if $\Pr(X \geq Y) = 1$.[2]

Risk measures satisfying all four of these properties are called *coherent*.

Risk measures with variance in their formula (such as variance itself and semi-variance) fail the monotonicity property, since a constant has less variance than a random variable that varies, even if the random variable is always less than the constant.

Value-at-risk is not subadditive and therefore not coherent, but tail value-at-risk is coherent. Value-at-risk satisfies the other properties. In special cases, such as when all distributions under consideration are normal, value-at-risk is coherent.

Exercises

2.1. A project requires an immediate investment of 12 million and an additional investment of 1 million per year for 5 years starting at the end of year 1. The project will generate free cash flows (ignoring the investment cash flows) of 1.5 million in year 1, growing 2% per year perpetually. The cost of capital is 10%.

Calculate the NPV of this project.

2.2. A project to produce new widgets requires a $10 million investment paid immediately. Installing the machinery will take one year, during which time no widgets will be sold. It is expected that the sale of widgets will generate $2 million of free cash flows in year 2, growing $200,000 per year until year 11, at which time they will become obsolete and will not be sold any more.

The cost of capital is 10%.

Calculate the NPV of this project.

[2] The study note does mention that the inequalities for coherence for downside risks are reversed.

Exercises continue on the next page . . .

Table 2.1: Formula Summary

NPV

$$\text{NPV} = \sum_{n=0}^{\infty} \frac{\text{FCF}_n}{(1+r)^n}$$

where

FCF_n is free cash flow at time n
r is the cost of capital

If free cash flows are k at time 1 and grow at constant rate g, and the cost of capital is r, then their NPV is

$$\frac{k}{r-g} \tag{2.1}$$

Downside semi-variance:

$$\sigma_{SV}^2 = \mathbf{E}\Big[\min\big(0,(R-\mu)\big)^2\Big] \tag{2.3}$$

Sample downside semi-variance:

$$\hat{\sigma}_{SV}^2 = \frac{1}{n}\sum_{i=1}^{n} \min\big(0,(R_i-\bar{R})\big)^2 \tag{2.4}$$

Value-at-risk:

$$\text{VaR}_\alpha(X) = F_X^{-1}(\alpha) \tag{2.5}$$

TVaR for downside risk:

$$\text{TVaR}_\alpha(X) = \mathbf{E}[X \mid X < \text{VaR}_\alpha(X)] = \frac{\int_{-\infty}^{\text{VaR}_\alpha(X)} x f(x)\,\mathrm{d}x}{\alpha} \tag{2.6}$$

TVaR for upside risk:

$$\text{TVaR}_\alpha(X) = \mathbf{E}[X \mid X > \text{VaR}_\alpha(X)] = \frac{\int_{\text{Var}_\alpha(X)}^{\infty} x f(x)\,\mathrm{d}x}{1-\alpha} \tag{2.7}$$

2.3. A project to produce desks requires an investment of \$20 million immediately. The machinery will last for 7 years, at which point the project ends. You are given:

(i) The desks will sell for \$500 apiece.
(ii) The same number of desks will be sold each year.
(iii) There will be fixed costs of \$1 million per year, and the variable costs associated with manufacturing and selling the desks are \$200 apiece.
(iv) The revenues from selling the desks and the associated fixed and variable costs occur at the end of each year.
(v) The cost of capital is 12%.

Based on a break-even analysis, calculate the number of desks per year that must be sold.

Exercises continue on the next page . . .

2.4. A project requires an immediate investment of 8 million. An additional investment of 2 million is required at the end of year 1. Starting in the second year, the project will generate free cash flows of 1 million per year, growing 3% per year perpetually.

Based on a break-even analysis, determine the cost of capital to break even.

2.5. A project requires an investment of 5 million. The following are baseline, best case, and worst case assumptions:

	Worst case	Baseline	Best case
Free cash flows in first year	1.1	1.2	1.3
Rate of growth of free cash flows	0	0.03	0.05
Number of years of free cash flows	7	10	13

The cost of capital is 0.10.

Which of the three assumptions in the table is the NPV most sensitive to?

2.6. A company invests 8 million in a project to produce a new product. The product can be perpetually. A sensitivity analysis considers the following assumptions:

	Worst case	Baseline	Best case
Annual number of units sold	1,000,000	1,200,000	1,500,000
Price per unit	1.25	1.50	1.60
Expenses, as percentage of sales price	23%	20%	15%

The cost of capital is 0.15.

To which assumption is the NPV most sensitive?

2.7. You are given the following sample:

$$1 \qquad 3 \qquad 7 \qquad 15 \qquad 25 \qquad 39$$

Calculate the downside semi-variance.

2.8. A random variable X follows a normal distribution with $\mu = 20$, $\sigma^2 = 100$.

Calculate the downside standard deviation of X.

2.9. A random variable X has the following probability density function:

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the downside semi-variance of X.

2.10. Profits X have the following cumulative distribution function:

$$F(x) = e^{-1000/x} \qquad x > 0$$

Calculate the value-at-risk at 1%.

Exercises continue on the next page . . .

2.11. Profits X have the following cumulative distribution function:

$$F(x) = \begin{cases} 1 - \left(\frac{1000}{x}\right)^2 & x > 1000 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the value-at-risk at 0.5%.

2.12. Losses on an insurance are distributed as follows:

Greater than	Less than or equal to	Probability
0	1000	0.45
1000	2000	0.25
2000	5000	0.22
5000	10000	0.05
10000	20000	0.03

Within each range losses are uniformly distributed.

Calculate the tail value-at-risk for losses at 95%.

2.13. Profits X have the following cumulative distribution function:

$$F(x) = 1 - e^{-x/1000} \qquad x > 0$$

Calculate the tail value-at-risk at 5%.

2.14. For a simulation with 100 runs, the largest 20 values are

920	920	922	925	926	932	939	940	943	945
948	952	959	962	969	976	989	1005	1032	1050

Estimate TVaR at 95% from this sample.

2.15. Consider the risk measure $g(X) = \mathbf{E}[X^2]$. Assume it is used only for nonnegative random variables. Which coherence properties does it satisfy?

2.16. Consider the risk measure $g(X) = \mathbf{E}[\sqrt{X}]$. Assume it is used only for nonnegative random variables. Which coherence properties does it satisfy?

Finance and Investment sample questions: 27,34,35,42

Solutions

2.1. The present value of the investment is $12 + (1 - 1/1.1^5)/0.1 = 15.791$ million. The present value of the free cash flows is $1/(0.1 - 0.02) = 18.75$ million. The NPV is $18.75 - 15.791 = \boxed{\textbf{2.959}\text{ million}}$.

2.2. At time 1, the present value of the cash flows from the widgets is $1{,}800{,}000a_{\overline{10|}} + 200{,}000(Ia)_{\overline{10|}}$.

$$a_{\overline{10|}} = \frac{1 - 1/1.1^{10}}{0.1} = 6.144567$$
$$(Ia)_{\overline{10|}} = \frac{6.144567 - 10/1.1^{10}}{0.1} = 22.89134$$

So the present value of the cash flows at time 1 is 6.144567(1,800,000)+22.89134(200,000) = 15,638,489. Discounting to time 0 and subtracting the investment, the NPV is 15,638,489/1.1 − 10,000,000 = $\boxed{\textbf{\$4,216,808}}$.

2.3. Present value of investment and fixed expenses is

$$20 + a_{\overline{7}|} = 20 + \frac{1 - 1/1.12^7}{0.12} = 24.563757 \text{ million}$$

Present value of net profit from sale of 1 desk per year is

$$300\left(\frac{1 - 1/1.12^7}{0.12}\right) = 1369$$

So to break even, 24,563,757/1369 = $\boxed{\textbf{17,941}}$ desks per year must be sold.

2.4. Let r be the cost of capital. At time 1, the present value of future free cash flows is $1/(r - 0.03)$ in millions. Thus at time 0 the present value of these cash flows is $1/\big((1+r)(r-0.03)\big)$. We want r such that

$$\begin{aligned}
-8 - \frac{2}{1+r} + \frac{1}{(1+r)(r-0.03)} &= 0\\
-8(1+r)(r-0.03) - 2(r-0.03) + 1 &= 0\\
8r^2 + 9.76r - 1.3 &= 0\\
r &= \boxed{\mathbf{0.121163}}
\end{aligned}$$

2.5. The present value at annual effective rate r of cash flows for n years at the end of each year starting at 1 and growing at a rate of g is

$$\frac{1}{1+r}\sum_{k=0}^{n-1}\left(\frac{1+g}{1+r}\right)^k = \frac{1}{1+r}\,\frac{1 - \left(\frac{1+g}{1+r}\right)^n}{1 - (1+g)/(1+r)} = \frac{1 - \left(\frac{1+g}{1+r}\right)^n}{r - g}$$

In the following, all numbers are in millions. For the free cash flows in first year assumption, the NPVs of the worst and best cases are:

$$-8 + 1.1\left(\frac{1 - (1.03/1.1)^{10}}{0.10 - 0.03}\right) = -0.42788$$
$$-8 + 1.3\left(\frac{1 - (1.03/1.1)^{10}}{0.10 - 0.03}\right) = 0.94887$$

with difference 1.37675.

For the rate of growth assumption, the NPVs of the worst and best cases are:

$$-8 + 1.2\left(\frac{1 - (1/1.1)^{10}}{0.10}\right) = -0.62652$$
$$-8 + 1.2\left(\frac{1 - (1.05/1.1)^{10}}{0.10 - 0.05}\right) = 0.92777$$

with difference 1.55429

For the number of years of free cash flows assumption, the NPVs of the worst and best cases are:

$$-8 + 1.2\left(\frac{1 - (1.03/1.1)^{7}}{0.10 - 0.03}\right) = -1.67634$$
$$-8 + 1.2\left(\frac{1 - (1.03/1.1)^{13}}{0.10 - 0.03}\right) = 1.85060$$

with difference 3.52694.

The most sensitive assumption is **number of years**.

2.6. We will ignore the investment cost, which is the same in all scenarios.

For annual units sold, the NPVs of the worst and best cases are:

$$\frac{1{,}000{,}000(1.50)(0.8)}{0.15} = 8{,}000{,}000$$
$$\frac{1{,}500{,}000(1.50)(0.8)}{0.15} = 12{,}000{,}000$$

with difference 4,000,000.

For price per unit, the NPVs of the worst and best cases are:

$$\frac{1{,}200{,}000(1.25)(0.8)}{0.15} = 8{,}000{,}000$$
$$\frac{1{,}200{,}000(1.60)(0.8)}{0.15} = 10{,}240{,}000$$

with difference 2,240,000.

For expenses, the NPVs of the worst and best cases are:

$$\frac{1{,}200{,}000(1.50)(0.77)}{0.15} = 9{,}240{,}000$$
$$\frac{1{,}200{,}000(1.50)(0.85)}{0.15} = 10{,}200{,}000$$

with difference 960,000.

Annual units sold has the highest sensitivity.

2.7.

$$\bar{x} = \frac{1+3+7+15+25+39}{6} = 15$$
$$\sigma_{sv}^2 = \frac{(1-15)^2+(3-15)^2+(7-15)^2}{6} = \boxed{67\tfrac{1}{3}}$$

2.8. A normal distribution is symmetric. So the downside semi-variance and the upside semi-variance are equal, and the downside semi-variance is therefore half the upside semi-variance, or 50. The downside standard deviation is $\sqrt{50} = \boxed{7.0711}$.

2.9. This is a beta distribution. If you recognize it and are familiar with beta, you know that the mean is 2/3. Otherwise it is not hard to calculate:

$$\mathbf{E}[X] = \int_0^1 2x^2 dx = \frac{2}{3}$$

The downside semi-variance is

$$\begin{aligned}
\sigma_{sv}^2 &= \int_0^{2/3}\left(x-\frac{2}{3}\right)^2 2x\,\mathrm{d}x \\
&= \int_0^{2/3} 2x^3\,\mathrm{d}x - \int_0^{2/3}\frac{8}{3}x^2\,\mathrm{d}x + \int_0^{2/3}\frac{8}{9}x\,\mathrm{d}x \\
&= \frac{2}{4}\left(\frac{2}{3}\right)^4 - \frac{8/3}{3}\left(\frac{2}{3}\right)^3 + \frac{8/9}{2}\left(\frac{2}{3}\right)^2 = \boxed{0.032922}
\end{aligned}$$

2.10. We need the first percentile of profits. Let it be x. Then

$$\begin{aligned} e^{-1000/x} &= 0.01 \\ \frac{1000}{x} &= -\ln 0.01 \\ x &= -\frac{1000}{\ln 0.01} = \boxed{\mathbf{217.15}} \end{aligned}$$

2.11. Let x be the VaR. Then

$$\begin{aligned} \left(\frac{1000}{x}\right)^2 &= 0.995 \\ \frac{1000}{x} &= 0.997497 \\ x &= \frac{1000}{0.997497} = \boxed{\mathbf{1002.51}} \end{aligned}$$

2.12. The 95$^{\text{th}}$ percentile of losses is the point with a 5% probability of losses above that point. Since the top interval has probability 3%, we need a subset of the $(5000, 10000]$ interval with probability 2%. That interval has probability 5%, so we need the top $2/5$ of the interval, making 8000 the 95$^{\text{th}}$ percentile. The expected value of losses given that they're above 8000 can be calculated using the double expectation formula:

$$\begin{aligned} \text{TVaR}(X) &= \mathbf{E}[X \mid X > 8000] \\ &= \Pr(X \le 10000 \mid X > 8000)\,\mathbf{E}[X \mid 8000 < X \le 10000] + \Pr(X > 10000)\,\mathbf{E}[X \mid X > 10000] \end{aligned}$$

By uniformity, $\mathbf{E}[X \le 10000 \mid X > 8000] = 9000$ and $\mathbf{E}[X \mid X > 10000] = 15000$. So

$$\text{TVaR}(X) = 0.4(9000) + 0.6(15000) = \boxed{\mathbf{12{,}600}}$$

2.13. The 5$^{\text{th}}$ percentile of X is

$$\begin{aligned} e^{-x/1000} &= 0.95 \\ x &= -1000 \ln 0.95 = 51.2933 \end{aligned}$$

The straightforward way to calculate the conditional expectation is to integrate x over the density function and then divide by 0.05, the probability of $X < 51.2933$. The density function is $0.001e^{-x/1000}$.

$$\begin{aligned} \int_0^{51.2933} 0.001xe^{-x/1000}\,\mathrm{d}x &= -xe^{-x/1000}\Big|_0^{51.2933} + \int_0^{51.2933} e^{-x/1000}\,\mathrm{d}x \\ &= -51.2933e^{-0.0512933} - 1000e^{-0.0512933} + 1000 = 1.27137 \\ \text{TVaR}_{0.01}(X) &= \frac{1.27137}{0.05} = \boxed{\mathbf{25.4274}} \end{aligned}$$

2.14. Average the top 5 numbers.

$$\frac{976 + 989 + 1005 + 1032 + 1050}{5} = \boxed{\mathbf{1010.4}}$$

2.15. $g(X)$ does not satisfy translation invariance since $\mathbf{E}[(X+c)^2] \neq \mathbf{E}[X]^2 + c$. It does not satisfy positive homogeneity since $\mathbf{E}[(cX)^2] \neq c\,\mathbf{E}[X^2]$. Also, $\mathbf{E}[(X+Y)^2] = \mathbf{E}[X^2] + 2\,\mathbf{E}[XY] + \mathbf{E}[Y^2]$. Since $\mathbf{E}[XY]$ may be greater than 0, it does not satisfy subadditivity. However, it does satisfy monotonicity, since if $X \leq Y$, and both X and Y are nonnegative, then $X^2 - Y^2 \leq 0$ so $\mathbf{E}[X^2] - \mathbf{E}[Y^2] \leq 0$.

2.16. $g(X)$ does not satisfy translation invariance since $\mathbf{E}\left[\sqrt{X+c}\right] \neq \mathbf{E}\left[\sqrt{X}\right] + c$. It does not satisfy positive homogeneity since $\mathbf{E}\left[\sqrt{cX}\right] \neq c\,\mathbf{E}[\sqrt{X}]$. For subadditivity, we need

$$\mathbf{E}\left[\sqrt{X+Y}\right] \leq \mathbf{E}\left[\sqrt{X}\right] + \mathbf{E}\left[\sqrt{Y}\right] = \mathbf{E}\left[\sqrt{X} + \sqrt{Y}\right]$$

This will be true if $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for all $x, y \geq 0$. Square both sides of the inequality, and this is equivalent to $\sqrt{xy} \geq 0$, which is true. So this risk measure is subadditive. It is also monotonic, since if $X \leq Y$, then $\sqrt{X} - \sqrt{Y} \leq 0$ and therefore the expected value of $\sqrt{X} - \sqrt{Y}$ is greater than 0.

Quiz Solutions

2-1. The present value of the free cash flows, in millions, is $1/(0.09 - 0.03) = 16.666667$. The NPV is $16{,}666{,}667 - 10{,}000{,}000 = \boxed{\mathbf{6{,}666{,}667}}$.

2-2. Let n be the number of years the product sells. The NPV in millions is $-9 + a_{\overline{n}|}$.

$$\begin{aligned}
-9 + \frac{1 - 1/1.1^n}{0.1} &= 0\\
1 - \frac{1}{1.1^n} &= 0.9\\
1.1^n = \frac{1}{0.1} &= 10\\
n \ln 1.1 &= \ln 10\\
n &= 24.1589
\end{aligned}$$

Since we have no provision for fractional years, the answer is $\boxed{\mathbf{25}}$, although this leads to a slightly positive NPV.

2-3. The mean is 15. The sample semi-variance is

$$\frac{(5-15)^2 + (10-15)^2}{5} = \boxed{\mathbf{25}}$$

Practice Exam 1

1. Options on XYZ stock trade on the Newark Exchange. Each option is for 100 shares. You are given that on March 21:

(i) 3000 options traded.
(ii) The price of a share of stock was \$40.
(iii) The price of each option was \$90.

Determine the total notional value of all of the options traded.

(A) 3,000 (B) 9,000 (C) 120,000 (D) 270,000 (E) 12,000,000

2. For American put options on a stock with identical expiry dates, you are given the following prices:

Strike price	Put premium
30	2.40
35	6.40

For an American put option on the same stock with the same expiry date and strike price 38, which of the following statements is correct?

(A) The lowest possible price for the option is 8.80.
(B) The highest possible price for the option is 8.80.
(C) The lowest possible price for the option is 9.20.
(D) The highest possible price for the option is 9.20.
(E) The lowest possible price for the option is 9.40.

3. A company has 100 shares of ABC stock. The current price of ABC stock is 30. ABC stock pays no dividends.

The company would like to guarantee its ability to sell the stock at the end of six months for at least 28.

European call options on the same stock expiring in 6 months with exercise price 28 are available for 4.10.

The continuously compounded risk-free interest rate is 5%.

Determine the cost of the hedge.

(A) 73 (B) 85 (C) 99 (D) 126 (E) 141

4. You are given the following prices for a stock:

Time	Price
Initial	39
After 1 month	39
After 2 months	37
After 3 months	43

A portfolio of 3-month Asian options, each based on monthly averages of the stock price, consists of the following:

(i) 100 arithmetic average price call options, strike 36.
(ii) 200 geometric average strike call options.
(iii) 300 arithmetic average price put options, strike 41.

Determine the net payoff of the portfolio after 3 months.

(A) 1433 (B) 1449 (C) 1464 (D) 1500 (E) 1512

5. The price of a 6-month futures contract on widgets is 260.

A 6-month European call option on the futures contract with strike price 256 is priced using Black's formula.

You are given:

(i) The continuously compounded risk-free rate is 0.04.
(ii) The volatility of the futures contract is 0.25.

Determine the price of the option.

(A) 19.84 (B) 20.16 (C) 20.35 (D) 20.57 (E) 20.74

6. The beta of a stock is 0.8. The volatility of the stock is 0.3.

The volatility of the market portfolio is 0.2.

Calculate the non-diversifiable risk of the stock, as measured by volatility.

(A) 0.16 (B) 0.20 (C) 0.25 (D) 0.40 (E) 0.60

7. Investor A bought a 40-strike European call option expiring in 1 year on a stock for 5.50. Investor A earned a profit of 6.44 at the end of the year.

Investor B bought a 45-strike European call option expiring in 1 year on the same stock at the same time, and earned a profit of 3.22 at the end of the year.

The continuously compounded risk-free interest rate is 2%.

Determine the price of the 45-strike European call option.

(A) 3.67 (B) 3.71 (C) 3.75 (D) 3.78 (E) 3.82

Exam questions continue on the next page . . .

8. Which of the following are inconsistent with the semi-strong form of the efficient market hypothesis but not with the weak form?

I. Changes in a stock's price in a week are positively correlated with changes in that stock's price in the previous week.

II. One can beat the market by using publicly available information.

III. One can beat the market by using hard-to-get information.

(A) None (B) I only (C) II only (D) III only
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

9. You own 100 shares of a stock whose current price is 42. You would like to hedge your downside exposure by buying 100 6-month European put options with a strike price of 40. You are given:

(i) The Black-Scholes framework is assumed.
(ii) The continuously compounded risk-free interest rate is 5%.
(iii) The stock pays no dividends.
(iv) The stock's volatility is 22%.

Determine the cost of the put options.

(A) 121 (B) 123 (C) 125 (D) 127 (E) 129

10. You are given the following information for a European call option expiring at the end of three years:

(i) The current price of the stock is 66.
(ii) The strike price of the option is 70.
(iii) The continuously compounded risk-free interest rate is 0.05.
(iv) The continuously compounded dividend rate of the stock is 0.02.

The option is priced using a 1-period binomial tree with $u = 1.3$, $d = 0.7$.

A replicating portfolio consists of shares of the underlying stock and a loan.

Determine the amount borrowed in the replicating portfolio.

(A) 14.94 (B) 15.87 (C) 17.36 (D) 17.53 (E) 18.43

11. A company has a 25% probability of having 50 million in assets and a 75% probability of having 150 million in assets at the end of one year. It has debt of 80 million due in one year. Bankruptcy costs are 20 million.

The cost of debt capital is 6% and the cost of equity capital is 18%.

The corporate tax rate is 20%.

Calculate the value of the company.

(A) 101.1 million (B) 108.1 million (C) 108.9 million (D) 112.8 million (E) 113.7 million

Exam questions continue on the next page . . .

12. For European options on a stock having the same expiry and strike price, you are given:

(i) The stock price is 85.
(ii) The strike price is 90.
(iii) The continuously compounded risk free rate is 0.04.
(iv) The continuously compounded dividend rate on the stock is 0.02.
(v) A call option has premium 9.91.
(vi) A put option has premium 12.63.

Determine the time to expiry for the options.

(A) 3 months (B) 6 months (C) 9 months (D) 12 months (E) 15 months

13. A portfolio of European options on a stock consists of a bull spread of calls with strike prices 48 and 60 and a bear spread of puts with strike prices 48 and 60.

You are given:

(i) The options all expire in 1 year.
(ii) The current price of the stock is 50.
(iii) The stock pays dividends at a continuously compounded rate of 0.01.
(iv) The continuously compounded risk-free interest rate is 0.05.

Calculate the price of the portfolio.

(A) 9.51 (B) 9.61 (C) 9.90 (D) 11.41 (E) 11.53

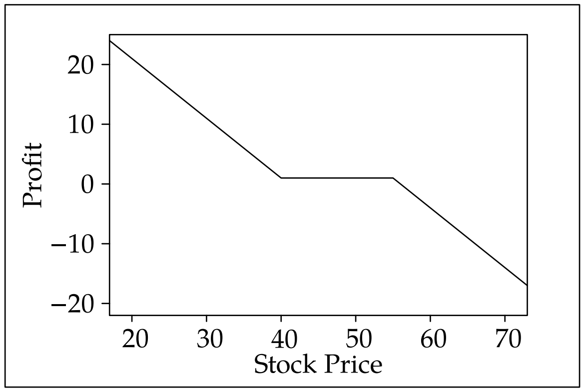
14. Stock ABC's expected annual rate of return is 0.10 with volatility 0.25. Stock DEF's expected annual rate of return is 0.08 with volatility 0.30. An equally weighted portfolio of the two stocks has a volatility of 0.22.

Calculate the correlation between the two stock returns.

(A) 0.021 (B) 0.079 (C) 0.137 (D) 0.274 (E) 0.548

Exam questions continue on the next page . . .

15. You are given the following graph of the profit on a position with derivatives:



Determine which of the following positions has this profit graph.

(A) Long forward
(B) Short forward
(C) Long collar
(D) Long collared stock
(E) Short collar

16. Which of the following behaviors may make the market portfolio inefficient?

I. Investors invest in stocks they are most familiar with.

II. Investors seek sensation.

III. Investors hang on to losers and sell winners.

(A) I and II only (B) I and III only (C) II and III only (D) I, II, and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

17. For a put option on a stock:

(i) The premium is 2.56.
(ii) Delta is −0.62.
(iii) Gamma is 0.09.
(iv) Theta is −0.02 per day.

Calculate the delta-gamma-theta approximation for the put premium after 3 days if the stock price goes up by 2.

(A) 1.20 (B) 1.32 (C) 1.44 (D) 1.56 (E) 1.62

18. Which of the following are differences between a venture capital firm and a private equity firm?

I. Private equity firms deal with existing privately held firms while venture capital firms deal with start-up companies.

II. Private equity deals are larger.

III. General partners of venture capital firms receive carried interest, unlike general partners of private equity firms.

(A) I and II (B) I and III (C) II and III (D) I, II, and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

19. For an at-the-money European call option on a nondividend paying stock:

(i) The price of the stock follows the Black-Scholes framework
(ii) The option expires at time t.
(iii) The option's delta is 0.5832.

Calculate delta for an at-the-money European call option on the stock expiring at time $2t$.

(A) 0.62 (B) 0.66 (C) 0.70 (D) 0.74 (E) 0.82

20. You are given the following sample:

10 25 48 52 100 125

Calculate the sample downside standard deviation.

(A) 25.4 (B) 25.6 (C) 27.0 (D) 27.4 (E) 27.8

21. Gap options on a stock have six months to expiry, strike price 50, and trigger 49. You are given:

(i) The stock price is 45.
(ii) The continuously compounded risk free rate is 0.08.
(iii) The continuously compounded dividend rate of the stock is 0.02.

The premium for a gap call option is 1.68.

Determine the premium for a gap put option.

(A) 4.20 (B) 5.17 (C) 6.02 (D) 6.96 (E) 7.95

22. Determine which of the following positions has the same cash flow as a short zero-coupon bond position.

(A) Long stock and long forward
(B) Long stock and short forward
(C) Short stock and long forward
(D) Short stock and short forward
(E) Long forward and short forward

Exam questions continue on the next page . . .

23. A 1-year American pound-denominated put option on euros allows the sale of €100 for £90. It is modeled with a 2-period binomial tree based on forward prices. You are given

(i) The spot exchange rate is £0.8/€.
(ii) The continuously compounded risk-free rate in pounds is 0.06.
(iii) The continuously compounded risk-free rate in euros is 0.04.
(iv) The volatility of the exchange rate of pounds to euros is 0.1.

Calculate the price of the put option.

(A) 8.92 (B) 9.36 (C) 9.42 (D) 9.70 (E) 10.00

24. You are conducting a break-even analysis on a project. The project has the following parameters:

(i) Initial investment: 16 million.
(ii) Free cash flows in first year: 2 million.
(iii) Rate of growth in cash flows: 3% per year.
(iv) Cost of capital: 12% annual effective rate.
(v) Project lifetime: infinite

Calculate the cost of capital to break even.

(A) 0.115 (B) 0.125 (C) 0.135 (D) 0.145 (E) 0.155

25. The price of an asset, $X(t)$, follows the Black-Scholes framework. You are given that

(i) The continuously compounded expected rate of appreciation is 0.1.
(ii) The volatility is 0.2.

Determine $\Pr\left(X(2)^3 > X(0)^3\right)$.

(A) 0.63 (B) 0.65 (C) 0.67 (D) 0.69 (E) 0.71

26. A market-maker writes a 1-year call option and delta-hedges it. You are given:

(i) The stock's current price is 100.
(ii) The stock pays no dividends.
(iii) The call option's price is 4.00.
(iv) The call delta is 0.76.
(v) The call gamma is 0.08.
(vi) The call theta is −0.02 per day.
(vii) The continuously compounded risk-free interest rate is 0.05.

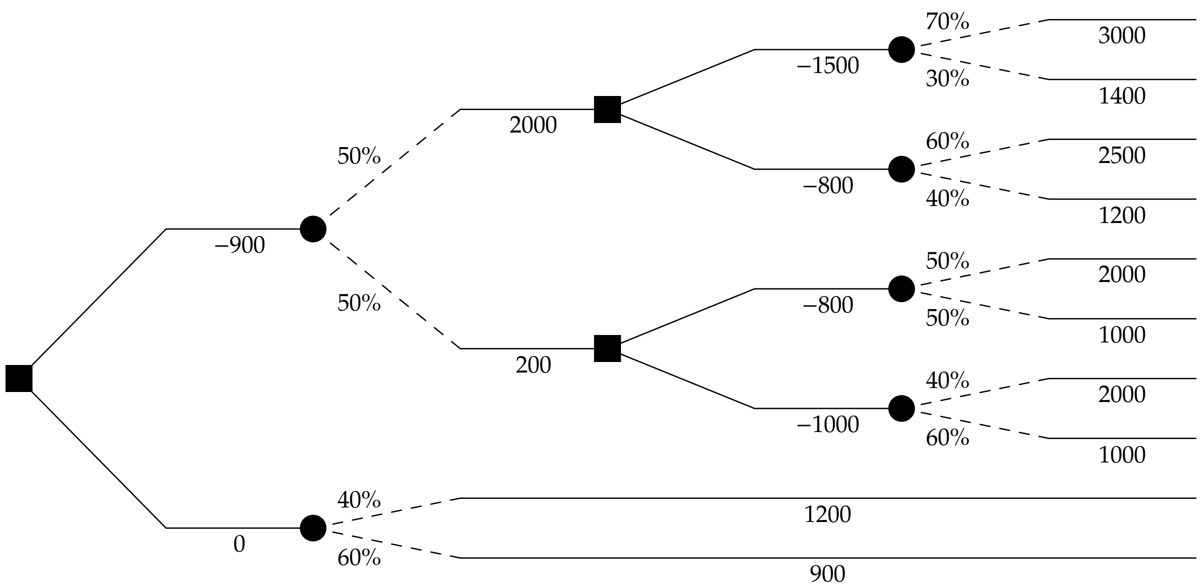
The stock's price rises to 101 after 1 day.

Estimate the market-maker's profit.

(A) −0.04 (B) −0.03 (C) −0.02 (D) −0.01 (E) 0

Exam questions continue on the next page . . .

27. You are given the following decision tree for a project:



The interest rate is 0.

Calculate the expected profit using the optimal strategy.

(A) 1000 (B) 1020 (C) 1040 (D) 1060 (E) 1080

28. You are given:

(i) The price of a stock is 40.
(ii) The continuous dividend rate for the stock is 0.02.
(iii) Stock volatility is 0.3.
(iv) The continuously compounded risk-free interest rate is 0.06.

A 3-month at-the-money European call option on the stock is priced with a 1-period binomial tree. The tree is constructed so that the risk-neutral probability of an up move is 0.5 and the ratio between the prices on the higher and lower nodes is $e^{2\sigma\sqrt{h}}$, where h is the amount of time between nodes in the tree.

Determine the resulting price of the option.

(A) 3.11 (B) 3.16 (C) 3.19 (D) 3.21 (E) 3.28

29. You are given the following information for two stocks:

	Expected return	Volatility
Stock A	0.2	0.3
Stock B	0.1	0.2

The correlation between the two stocks is -0.5.

A portfolio consists of 16% Stock A and 84% Stock B.

There is a more efficient portfolio of the two stocks having the same volatility.

Determine the proportion of that portfolio invested in Stock A.

(A) 0.44 (B) 0.52 (C) 0.58 (D) 0.62 (E) 0.68

30. For a portfolio of call options on a stock:

Number of shares of stock	Call premium per share	Delta
100	11.4719	0.6262
100	11.5016	0.6517
200	10.1147	0.9852

Calculate delta for the portfolio.

(A) 0.745 (B) 0.812 (C) 0.934 (D) 297.9 (E) 324.8

Solutions to the above questions begin on page **??**.

Appendix A. Solutions to Old Exams

Note: At the time the SOA Spring 2007 and Spring 2009 exams were given, the SOA had specific rules for use of the cumulative normal distribution function. These rules stipulated that no interpolation was to be done in the printed normal distribution tables. Instead, for evaluating the normal distribution function, the argument is rounded to two decimal places. For evaluating the inverse of the normal distribution function, use the inverse of the value in the table nearest the argument. When these rounding rules lead to a different answer, the following solutions do it both with the old rounding rules and with 5-decimal place precision.

A.1 Solutions to SOA Exam MFE, Spring 2007

The questions for this exam may be downloaded from

`http://www.soa.org/files/edu/edu-mc-exam-mfe-0507.pdf`

1. [**Lesson 18**] By put-call parity, with D equal to one dividend

$$\begin{aligned}
P(52,50,0.5)-C(52,50,0.5) &= 50e^{-0.06(0.5)}-\big(52-\mathrm{PV(Dividends)}\big)\\
2.45-4.50 &= 50e^{-0.03}-52+D\left(e^{-0.01}+e^{-0.025}\right)\\
-2.05-50e^{-0.03}+52 &= D\left(e^{-0.01}+e^{-0.025}\right)\\
D &= \frac{52-2.05-48.5223}{1.9654}=\boxed{\mathbf{0.726}}\qquad\textbf{(B)}
\end{aligned}$$

2. *Question 2 is not on the current Exam IFM syllabus*

3. [**Lesson 24**] Use the Black-Scholes formula. When this exam was given, they expected you to use rounded values of the normal CDF, as follows:

$$\begin{aligned}
d_1 &= \frac{\ln(100/98)+\big(0.055-0.01+0.5(0.5^2)\big)(0.5)}{0.5\sqrt{0.5}}=0.29756\\
d_2 &= 0.29756-0.5\sqrt{0.5}=-0.05600\\
N(-d_1) &= N(-0.30)=0.3821\\
N(-d_2) &= N(0.06)=0.5239
\end{aligned}$$

$$P(100,98,0.50,0.055,0.5,0.01)=98e^{-0.055(0.5)}(0.5239)-100e^{-0.01(0.5)}(0.3821)=\boxed{\mathbf{11.93}}\qquad\textbf{(C)}$$

Using 5-digit precision values:

$$\begin{aligned}
N(-d_1) &= N(-0.29756)=0.38302\\
N(-d_2) &= N(0.05600)=0.52233
\end{aligned}$$

$$P(100,98,0.50,0.055,0.5,0.01)=98e^{-0.055(0.5)}(0.52233)-100e^{-0.01(0.5)}(0.38202)=\boxed{\mathbf{11.689}}$$

4. [**Lesson 22**] Exercise of a put is rational only if the present value of interest on the strike price exceeds the present value of future dividends. The present value of future dividends on the stock is $50(1-e^{-0.08})$ and the present value of interest on the strike price is $K(1-e^{-0.04})$. In order for the latter to exceed the former, $K > 50(1-e^{-0.08})/(1-e^{-0.04}) = 98.04$. This doesn't even consider other benefits of holding the put option, namely the implicit call option if the stock price rises. Thus it is never optimal to exercise the put option for the choices given. **(E)**

A harder way to do the problem is to use put-call parity to calculate the put value and verify that it is greater than $K - S$. Clearly $K = 40$ and $K = 50$, when the put has no value, are not optimal. $Se^{-\delta} = 50e^{-0.08} = 46.16$. So for the two strikes 60 and 70, we have:

$$P(50, 60, 1) = 0.71 + 60e^{-0.04} - 46.16 = 12.20 > 60 - 50$$
$$P(50, 70, 1) = 0 + 70e^{-0.04} - 46.16 = 21.10 > 70 - 50$$

showing that exercise is not optimal for these strikes.

5. [**Lesson 25**] The volatility of the call option is Ω times the volatility of the stock.

$$\begin{aligned}
d_1 &= \frac{\ln(85/80) + 0.055 + 0.5(0.5^2)}{0.5} = 0.48125 \\
d_2 &= 0.481 - 0.5 = -0.01875 \\
N(d_1) &= N(0.48125) = 0.68483 \\
N(d_2) &= N(-0.01875) = 0.49252 \\
\Delta &= N(d_1) = 0.68483 \\
C(85, 80, 0.50, 0.055, 1, 0) &= 85(0.6844) - 80e^{-0.055}(0.49252) = 20.9175 \\
\Omega &= \frac{S\Delta}{C} = \frac{85(0.6844)}{20.9175} = 2.7829 \\
\sigma_{\text{call}} &= 0.5(2.7829) = \boxed{1.39} \qquad \textbf{(D)}
\end{aligned}$$

6. [**Lesson 28**] (iii) and (iv) are fancy ways of saying that the continuous dividend rate on the stocks are 0.05 and 0.10. You don't really need to use (iii).

The option in (v) pays $\max\bigl(S_1(3) - S_2(3), 0\bigr)$. If we add $S_2(3)$ to this, we have $\max\bigl(S_1(3), S_2(3)\bigr)$. So the claim is currently worth 10 (the price of the option) plus the present value of $S_2(3)$, which is $S_2e^{-3\delta} = 200e^{-0.3} = 148.16$. The total value of the claim is $148.16 + 10 = \boxed{158.16}$. **(C)**

7. *Question 7 is not on the current Exam IFM syllabus*

8. [**Lesson 24**] The variance of $\ln S(t)$ is the square of the volatility. Since $K = Se^{rT}$, $Ke^{-rT} = S$ and we don't need r. In fact, $\ln(S/K) + r = 0$, so the Black-Scholes formula becomes

$$\begin{aligned}
d_1 &= \frac{0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{0.5(0.4)(10)}{\sqrt{0.4(10)}} = 1 \\
d_2 &= 1 - \sqrt{0.4(10)} = -1 \\
N(d_1) &= N(1) = 0.84134 \\
N(d_2) &= N(-1) = 0.15866 \\
C &= 100(0.84134) - 100(0.15866) = \boxed{68.268} \qquad \textbf{(C)}
\end{aligned}$$

When the exam was given, rounding rules resulted in rounding $N(d_1)$ and $N(d_2)$ to four decimal places, making the answer 68.26.

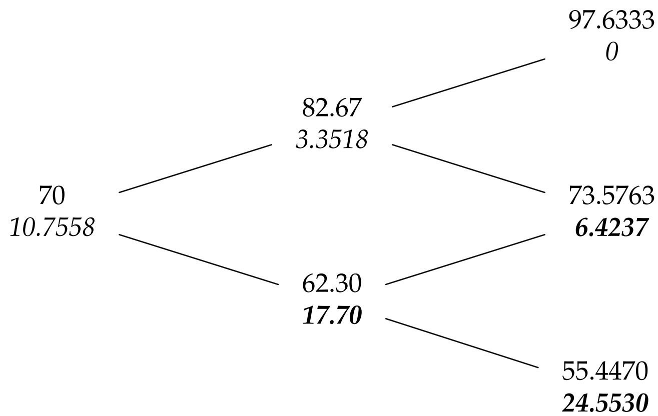


Figure A.1: Binomial tree for question 11

9. *Question 9 is not on the current Exam IFM syllabus*

10. [**Section 26.3**] Vega is extraneous. Since gamma for the stock is 0, to match gammas we need (letting x_2 be the amount of Call-II to buy)

$$-1000(0.0651) + x_2(0.0746) = 0$$
$$x_2 = \frac{65.1}{0.0746} = \boxed{\mathbf{872.7}}$$

making **(B)** the only possible choice. Just to make sure, you may solve for the amount of stock, x_0, by matching deltas.

$$x_0 + 872.7(0.7773) - 1000(0.5825) = 0$$
$$x_0 = 582.5 - 872.7(0.7773) = \boxed{\mathbf{-95.8}}$$

so we sell 95.8 shares of stock.

11. [**Lesson 21**] The risk-neutral probability is

$$p^* = \frac{e^{0.025} - 0.89}{1.181 - 0.89} = 0.465$$

Figure A.1 shows the values of the stock and the option at all nodes. At the d node, exercising the put is optimal, because the holding value is

$$e^{-0.025}\big(0.465(6.4237) + 0.535(24.5530)\big) = 15.7248$$

which is less than the exercise value of $80 - 62.30 = 17.70$. The initial value is computed as

$$P = e^{-0.025}\big(0.465(3.3518) + 0.535(17.70)\big) = \boxed{\mathbf{10.7558}} \qquad \textbf{(D)}$$

12–13. *Questions 12–13 are not on the current Exam IFM syllabus*

14. [**Lesson 20**] $u = 70/60$, $d = 45/60 = 0.75$, so

$$p^* = \frac{e^{0.08} - 0.75}{70/60 - 0.75} = 0.8$$

The payoff is 20 at the upper node, 5 at the lower node, so its value is

$$C = e^{-0.08}\big(0.8(20) + 0.2(5)\big) = \boxed{\mathbf{15.69}} \qquad \textbf{(E)}$$

15. **[Lesson 24]** The prepaid forward price for the stock is $50 - 1.5e^{-0.05/3} = 48.5248$. Then

$$\begin{aligned}
d_1 &= \frac{\ln(48.5248/50) + \big(0.05 + 0.5(0.3^2)\big)(0.5)}{0.3\sqrt{0.5}} = 0.08274\\
d_2 &= 0.08274 - 0.3\sqrt{0.5} = -0.12939\\
N(-d_1) &= N(-0.08274) = 0.46703\\
N(-d_2) &= N(-0.12939) = 0.55148\\
P(S,K,t) &= 50e^{-0.025}(0.55148) - 48.5248(0.46703) = \boxed{\mathbf{4.231}}
\end{aligned}$$

With the rounding rules for the normal CDF in effect when this exam was given, the calculations would be

$$\begin{aligned}
N(-d_1) &= N(-0.08) = 0.4681\\
N(-d_2) &= N(0.13) = 0.5517\\
P(S,K,t) &= 50e^{-0.025}(0.5517) - 48.5248(0.4681) = \boxed{\mathbf{4.189}} \qquad \textbf{(C)}
\end{aligned}$$

16. *Question 16 is not on the current Exam IFM syllabus*

17. **[Lesson 28]** The gap option has strike price $K_1 = 90$ and trigger $K_2 = 100$. The price of a European call option is

$$Se^{-\delta t}N(d_1) - Ke^{-rt}N(d_2) = 80(0.2) - 100e^{-rt}N(d_2)$$

since $\Delta = e^{-\delta t}N(d_1)$. We're given that this equals 4, so

$$\begin{aligned}
16 - 100e^{-rt}N(d_2) &= 4\\
e^{-rt}N(d_2) &= 0.12
\end{aligned}$$

The price of the gap option is

$$Se^{-\delta t}N(d_1) - K_1e^{-rt}N(d_2) = 16 - 90(0.12) = \boxed{\mathbf{5.2}} \qquad \textbf{(B)}$$

18. *Question 18 is not on the current Exam IFM syllabus*

19. **[Section 26.2]** The change in stock price $\epsilon = 31.50 - 30.00 = 1.50$. The change in the option price is approximately

$$\Delta\epsilon + 0.5\Gamma\epsilon^2 = -0.28(1.50) + 0.5(0.10)(1.50^2) = -0.3075$$

so the new price is $4.00 - 0.3075 = \boxed{\mathbf{3.6925}}$. **(D)**

A.2 Solutions to CAS Exam 3, Spring 2007

The questions can be found at `http://www.casact.org/admissions/studytools/exam3/sp07-3.pdf`.

1–2. *Questions 1–2 are not on the current Exam IFM syllabus*

3. [**Lesson 18**] Under put-call parity, the difference in prices between the put and the call should be

$$\begin{aligned} P - C &= Ke^{-rt} - Se^{-\delta t} \\ &= 50e^{-0.03(0.5)} - 49.7e^{-0.02(0.5)} = 0.05012 \end{aligned}$$

The actual difference is 0.35, so buy buying a call and selling a put and doing the opposite of the right hand side of the above equation by lending Ke^{-rt} and selling $e^{-\delta t}$ shares of stock (which was calculated to cost 0.05012), we gain $0.35 - 0.05012 = \boxed{\mathbf{0.29988}}$. **(B)**

It is unclear when the question asks for the arbitrage per share whether they mean one call and one put or one share of stock. If they mean one share of stock, one would have to multiply all of the above transactions by $e^{\delta t} = 1.01005$ and would obtain the answer $0.29988(1.01005) = 0.30289$, leading to the same answer choice.

4. [**Lesson 18**] By put-call parity, $P = C + Ke^{-rt} - [S - \text{PV(Dividends)}]$. We have

$$\begin{aligned} &C = 2 \\ &Ke^{-rt} = 30e^{-0.05} = 28.54 \\ &S - \text{PV(Dividends)} = 29 - 0.5(e^{-0.2/12} + e^{-0.5/12}) = 28.03 \end{aligned}$$

Hence the put price is $2 + 28.54 - 28.03 = \boxed{\mathbf{2.51}}$. **(E)**

5–11. *Questions 5–11 are not on the current Exam IFM syllabus*

12. [**Lesson 19**]

I. American options allow everything European options allow and also allow early election, so they must be worth at least as much. ✓

II. For a put, the payoff is strike price minus stock price, so a higher stock price makes the value of the put *lower*. ✗

III. For a call, the payoff is stock price minus strike price, so a higher strike price makes the payoff and the value of the call *lower*. ✗

(A)

13. [**Lesson 18**] We are being asked for r. Use put-call parity.

$$\begin{aligned} P(85, 80, 0.25) - C(85, 80, 0.25) &= 80e^{-0.25r} - 85 \\ 1.60 - 6.70 &= 80e^{-0.25r} - 85 \\ 79.90 &= 80e^{-0.25r} \\ e^{-0.25r} &= \frac{79.90}{80} = 0.99875 \\ r &= -4\ln 0.99875 = \boxed{\mathbf{0.005003}} \qquad \textbf{(A)} \end{aligned}$$

14. **[Lesson 21]** First calculate the risk-neutral probability. Looking at the tree, we see that $u = 110/100 = 1.1$ and $d = 90/100 = 0.9$ and these are constant throughout the tree.

$$p^* = \frac{e^r - d}{u-d} = \frac{e^{0.05} - 0.9}{1.1 - 0.9} = 0.756355$$

The call pays off at the top 2 nodes. The probability of the top note is $p^{*2} = 0.756355^2 = 0.572074$ and the probability of the middle node is $2p^*(1-p^*) = 2(0.756355)(0.243645) = 0.368564$. The expected value of the payoff is

$$(121-95)(0.572074) + (99-95)(0.368564) = 16.3482$$

whose present value is $16.3482e^{-0.05(2)} = \boxed{\mathbf{14.79}}$. **(D)**

15. **[Lesson 20]** Although the question doesn't state it, a tree based on forward prices is assumed. Then

$$u = e^{(r-\delta)t+\sigma\sqrt{t}} = e^{(0.05-0.035)/3+0.3/\sqrt{3}} = 1.195070$$
$$d = e^{(r-\delta)t-\sigma\sqrt{t}} = e^{(0.05-0.035)/3-0.3/\sqrt{3}} = 0.845180$$
$$p^* = \frac{e^{(r-\delta)t} - d}{u-d} = \frac{1.005013 - 0.845180}{1.195070 - 0.845180} = \boxed{\mathbf{0.4568}} \qquad \textbf{(C)}$$

16. **[Lesson 20]** Δ is requested. We have that the call pays 6 at the upper node ($C_u = 6$) and 0 at the lower node ($C_d = 0$). The formula for Δ is

$$\Delta = e^{-\delta t}\frac{C_u - C_d}{S(u-d)} = \frac{6}{18-4} = \boxed{\mathbf{0.429}} \qquad \textbf{(C)}$$

17. **[Lesson 21]** The question assumes no dividends are paid and the tree is based on forward prices. u, d, and p^* are

$$u = e^{0.05+0.35} = 1.491824$$
$$d = e^{0.05-0.35} = 0.740818$$
$$p^* = \frac{e^{0.05} - 0.740818}{1.491824 - 0.740818} = 0.413382$$

Then $35d^2 = 35(0.740818^2) = 19.2084$, $35ud = 35(0.740818)(0.259182) = 6.7202$, and the option certainly doesn't pay off at the uu node. It only pays off at the dd node, where it pays $32 - 19.2084 = 12.7916$. The present value of the expected payoff is

$$P = (12.7916)(1-p^*)^2 e^{-0.05(2)} = (12.7916)(1-0.413382)^2(0.904837) = \boxed{\mathbf{3.98}} \qquad \textbf{(D)}$$

18–19. *Questions 18–19 are not on the current Exam IFM syllabus*

20. **[Lesson 24]** By Black-Scholes formula

$$d_1 = \frac{\ln(58.96/60) + \big(0.06 - 0.05 + 0.5(0.2^2)\big)0.25}{0.2\sqrt{0.25}} = -0.09985$$
$$d_2 = d_1 - 0.2\sqrt{0.25} = -0.19985$$
$$N(d_1) = N(-0.09985) = 0.46023$$
$$N(d_2) = N(-0.19985) = 0.42080$$
$$C(60, 58.96, 0.2, 0.06, 0.25, 0.05) = 58.96e^{-0.05(0.25)}(0.46023) - 60e^{-0.06(0.25)}(0.42080) = \boxed{\mathbf{1.926}} \qquad \textbf{(B)}$$

The expected annual return on the stock is irrelevant.