MLC: HOW TO SOLVE IT

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Insights & Shortcuts
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Preface

Rob is one of the smartest actuaries I have met. He told me that when he drives in Chicago, he typically follows the flow of the traffic, which means that he is driving at least 10 miles per hour faster than the speed limit. He said that if he drives under the speed limit he might be hit by a speeding car because everyone else is speeding. “In addition, why would I want time to tick away from me?” said Rob.

What hit me was his phrase “Why do I want time to tick away from me?” If you are reading this book, chances are that you a college student or a serious career switcher wanting to enter the actuarial profession. Or you are already an actuary wanting to get your ASA or FSA. No matter who you are, you are busy and you have no time to waste. You want to get the MLC done and move on with you life.

This is exactly the purpose of this book: to help you better prepare for MLC so you don’t waste a whole lot time.

My story

I came to U.S. when I was in my late twenties. I started my first corporate job in U.S. in the IT department of a large insurance company. After working there for about 2 years, I was ready to switch my career. If you have ever worked in a large IT department or any large department of a large company, you’ll find that there are thousands of people just like you who go to the same building in the morning around the same time you go to the building and who leave the same building in the afternoon around the same time you leave the building. The company’s giant parking lots were filled with thousands of cars, one of which was my second hand red Toyota Camry. The building is nice. Coworkers are nice. But I felt like a drop of water in the ocean.

I was floating around not sure how to make use of my life. Then one day I heard the actuary profession. I heard that if you were an actuary, you were among the elite group because there weren’t enough actuaries to go around. I was interested. I decided to study for P. By that time, I hadn’t touched calculus for 13 years. Fortunately, it took me just a couple of months to relearn calculus. I took P and got a 9. I was overjoyed. I applied for a job in the actuary department and became an actuary.

When I became an entry level actuary, I was in my early 30’s, about 8 years older than most of my peers, who got the actuary job straight from college. To quickly pass actuary exams, I used a bold strategy: reverse engineering. This is not for the faint of the heart. Think twice before you try it. It works like this. Before I took an exam, say MLC, I used a company printer and printed out all the released MLC exam papers and the official solution papers. There was a stack of paper on my desk. From the stack, I pulled out the most recent exam paper, looked up the SOA solutions, looked up the subject from the textbook, and studied the subject. Then I moved to the next exam paper. I call this just-in-time study, similar to the just-in-time inventory method used in Toyota and many other auto manufacturing plants around the world.

how to pass MLC or any actuary exam

Based on my experience of studying for actuary exams, I firmly believe that to pass an actuary exam you need to do 2 things: (1) you have to understand the core concepts, and (2) you have to be able to quickly solve the types of problems SOA likes to test.

Building a coherent body of knowledge of the subject matter is the most critical and the most time-consuming part of studying for an actuary exam. If you walk into the exam room muddleheaded or with scanty knowledge of basic theories, none of the tips or tricks you learned in an exam prep book would save you. Any chess master will tell you that there are no shortcuts in learning chess. You just have to know your stuff!

However, knowing the subject well doesn’t guarantee passing the exam or earning a high grade any more than good technical skills guarantee a job offer. It’s a sad reality that often those who know how to play the interview game get the job. When you take actuary exams, your knowledge is measured by your ability to solve the SOA style questions. To pass MLC, you’ll need to immerse yourself in the types of problems SOA likes to test or you’ll be one of those “theory smart, exam poor” people.

One key part of studying the SOA exam papers is to identify commonly tested problem types and learn how to solve them quickly. For example, finding the UL account value is tested in virtually every exam. Your first round of effort is to understand what is UL, what is Type A and Type B, what is COI, what is corridor, and how the UL account value builds up overtime. After you understand these basic concepts, you face a choice about how to find the Type A UL account value. Should you solve two linear equations on AV_t and COI_t or should you memorize the formulas for AV_t and COI_t to avoid having to solve two equations in the exam? You might try both approaches and see which method suits you. You might find that there’s no clear winner and that you want to learn both. However, before taking the exam, you must have a tried-and-true procedure for calculating the Type A UL account value. You don’t want to walk into exam empty handed without a proven method in your head.

Here’s the final point. It’s not absolutely necessary, but it helps. Most people’s performance will downgrade in the heat of the exam. To be safe, strive to learn at least a little bit more than the minimal knowledge required to pass MLC. When I was studying an old exam paper, I often asked myself “How can I make this problem harder?” If I saw a subject that was in the syllabus but that was not tested in the past, I often forced myself to learn at least a little bit about it. Even if the subject didn’t show up in the test, knowing that I was not a complete idiot on the subject reduced my anxiety.

know your stuff

Google Maps are handy especially when you go to a new place, but I hope you know how to get to your work or school when you left your phone at home or there's no internet connection. Over the years, I have developed many alternative routes for my daily commute. If route A is closed, I know what an alternative route to go. I know which road tends to be jammed by school buses, which road is more likely to have accidents, which alley is slippery when it snows. This knowledge serves me well. Just the other day, while I was driving to work, the road I took had a car accident. While most drivers were stuck in the traffic, I knew the exact small neighborhood that I needed to turn to bypass the traffic jam.

In virtually every career you choose, there's no substitute for learning the basics for doing the job. To study for MLC, you just have to learn the fundamentals: the force of mortality, the multiple state model, profit testing, to name a few.

a simple procedure beats the best mind

I remember a story I learned from a computer programming book. The story goes like this. A town in the Midwest has two coffee shops, A and B. If you visit Shop A, sometimes you can get coffee right away but other times they run out of coffee and you have to wait a little while. Shop B, on the other hand, always has coffee ready for a customer who just walks in. Both shops are in the same town and their workers have roughly the same skills. How does Shop B outsmart Shop A? In turns out that Shop B has a simple procedure. If you work in Shop B, from Day One you learn this rule: when existing coffee in a container reaches a certain low level, stop whatever you are doing and immediately start brewing new coffee. This procedure makes all the difference.

A procedure in programming is called an algorithm. When studying for an actuary exam, you'll need to build algorithms for commonly tested problems to avoid having to reinvent the wheel in the heat of the exam. When the big exam day comes, most of the problem types in the exam should be familiar to you and your job is just to recall pre-built algorithms. Don't purposely put yourself on the spot without an algorithm for finding the Type A UL account value. You have only several minutes per exam problem and in the heat of the exam it's really hard to invent a solution to an unseen problem type.

what you get from this book

In this book, I aim for 3 things:

- to teach you how to solve frequently tested problems. First and foremost, this is a problem solver book. I have no interest in regurgitating the AMLCR textbook. I throw you into problems right away and you either swim or sink. If you can solve my problems and understand my approach to problems, you are in good shape.

- to teach you core concepts. All my problems are designed to help you learn the fundamental concepts of MLC. I want you to understand, not just memorize. For example, if you understand that the two-term Woolhouse's formula is just the plain old trapezoidal rule, memorizing the Woolhouse's formula is far easier and more enjoyable. You'll get a headache if you swallow Woolhouse's formula without knowing why it works. With understanding comes newfound freedom and confidence.

- to challenge you a little bit. I purposely made many of my problems somewhat harder than the SOA problems, but my hard problems are still in the syllabus and are what you need to need. I want you to aim high.

acknowledgement

First, I want to thank two actuaries, Nathan Hardiman and Robin Cunningham, for their generosity. They gave me their Arch manual for the then Course 3 or Exam M practically free. Years ago they wrote a really good study manual called Arch for the then MLC. You might not know that of all the exams for ASA, MLC changes most frequently. For example, if you dig through old Course 3 exam papers, you'll find the famous problem of "lucky Tom finds coins at the Poisson rate of ... per hour." The Poisson distribution or Poisson process was a hot topic dreaded by many. To your relief, SOA dropped the Poisson process from the syllabus. Anyway, Nathan and Robin have their full time corporate actuarial jobs and couldn't keep up with frequent changes in the exam syllabus. Instead of withdrawing their Arch book from the market and letting their brain child die, they decided to give the Arch manual to another author. Since I wrote many study manuals, they gave me the book.

The Arch manual was a turning point for me technically. After downloading their manuscript from my email, I found out that Arch was written in LaTeX, not in Word. That was the first time I saw LaTeX code. At at time, I was looking for a solution to a long standing problem of Word crashing on me. Arch was a god sent. From Arch, I learned LaTeX and switched from Word to LaTeX for my future books.

In addition, I want to thank the many LaTeX contributors for their wonderful packages. Without LaTeX or many of its special packages, this book isn't possible.

Finally, I think you, dear reader, for reading the thoughts and reasoning I came up with after my actuarial day job. I hope you find this book useful. If you end up using this book, I thank you for the opportunity of being part of your journey into the actuarial dream land.

outlook of actuary profession

According to the U.S. Bureau of Labor Statistics, employment of actuaries is projected to grow 18% from 2014 to 2024, much faster than the average for all occupations. What are you waiting for? Study for MLC today!

FAQ

Does this book cover the entire syllabus?

Yes. The entire syllabus is covered.

Is this book sufficient for passing MLC?

No author can guarantee that if you read his book you will surely pass MLC. That said, if you can master this book and master the SOA exam papers, you have built a solid foundation for passing MLC.

What companion book do you recommend to use along side with this book?

You can use this book together with your favorite study guide. If you are not sure what other study guide to use, you can use this book together with the AMLC textbook and the SOA exam papers.

errata

Sample chapters and the errata for this book can be found at http://deeperunderstandingfastercalc.com/how2solveIt.php

Chapter 9

m-thly, UDD, W2, W3, W3*, claim acceleration

9.1 m-thly n-year term life insurance

The m-thly curtate future life time of (x)

$$K_x^{(m)} = \frac{\lfloor mK_x + m(T_x - K_x)\rfloor}{m} = K_x + \frac{\lfloor m(T_x - K_x)\rfloor}{m}$$

The PV random variable of an n-year term m-thly insurance of 1 on (x) is:

$$v^{\min\left(K_x^{(m)} + \frac{1}{m}, n\right)}$$

You should be able to calculate $A^{(12)}_{\overset{1}{x}:\overline{1}|}$ using the first principle.

Consider a 1-year term insurance of 1 on (x) that pays 1 at the end of the *month* of death. Its EPV is $A^{(12)}_{\overset{1}{x}:\overline{1}|}$. Now essentially we have a 12-month term insurance of 1 on (x).

$$A^{(12)}_{\overset{1}{x}:\overline{1}|} = A_{\overset{1}{x}:\overline{12}|j} = v_j\left({}_0p_x - {}_{\frac{1}{12}}p_x\right) + v_j^2\left({}_{\frac{1}{12}}p_x - {}_{\frac{2}{12}}p_x\right) + \ldots + v_j^{12}\left({}_{\frac{11}{12}}p_x - {}_{\frac{12}{12}}p_x\right)$$

$$j = (1+i)^{1/12} - 1 \text{ is the monthly interest rate and } v_j = 1/(1+j).$$

Similarly, an n-year term insurance of 1 on (x) that pays 1 at the end of the *month* of death is essentially a $12n$-monthly term insurance of 1 on (x). Its EPV is

$$A^{(12)}_{\overset{1}{x}:\overline{n}|} = A_{\overset{1}{x}:\overline{12n}|j} = v_j\left({}_0p_x - {}_{\frac{1}{12}}p_x\right) + v_j^2\left({}_{\frac{1}{12}}p_x - {}_{\frac{2}{12}}p_x\right) + \ldots + v_j^{12n}\left({}_{\frac{12n-1}{12}}p_x - {}_{\frac{12n}{12}}p_x\right)$$

Generally, $A^{(m)}_{\overset{1}{x}:\overline{n}|}$ is the EPV of an n-year term m-thly insurance of 1 on (x).

$$A^{(m)}_{\overset{1}{x}:\overline{n}|} = A_{\overset{1}{x}:\overline{mn}|j} = v_j\left({}_0p_x - {}_{\frac{1}{m}}p_x\right) + v_j^2\left({}_{\frac{1}{m}}p_x - {}_{\frac{2}{m}}p_x\right) + \ldots + v_j^{mn}\left({}_{\frac{mn-1}{m}}p_x - {}_{\frac{mn}{m}}p_x\right)$$

$$j = (+i)^{1/m} - 1 \text{ is the } m\text{-thly interest rate.}$$

$K_x^{(m)} = \dfrac{\lfloor mT_x\rfloor}{m}$ represents the m-thly curtate future life time of (x) and $\lfloor y\rfloor$ represents the greatest integer less than or equal to y.

Since $T_x = K_x + (T_x - K_x)$,

$$K_x^{(m)} = \frac{\lfloor mK_x + m(T_x - K_x)\rfloor}{m} = K_x + \frac{\lfloor m(T_x - K_x)\rfloor}{m}, \quad 0 \le T_x - K_x < 1$$

For example, for a 3-year term 12-thly insurance of 1 on (x), if $T_x = 2.65$, then

$$K_x^{(12)} = \frac{\lfloor 12(2.65)\rfloor}{12} = \frac{\lfloor 31.8\rfloor}{12} = \frac{31}{12} = 2\frac{7}{12}$$

,

$$Or \quad K_x^{(12)} = 2 + \frac{\lfloor 12(0.65)\rfloor}{12} = 2 + \frac{\lfloor 7.8\rfloor}{12} = 2 + \frac{7}{12}$$

So (x) has lived 2 full years and 7 full months and the death benefit of 1 will be paid at $2 + \dfrac{8}{12}$.

9.2 EPV: m-thly term insurance under UDD

You can skip-read the proof. Just remember the key formula:

$$UDD \Rightarrow \quad iA_{\overset{1}{x}:\overline{n}|} = i^{(m)} A^{(m)}_{\overset{1}{x}:\overline{n}|} = \delta \overline{A}_{\overset{1}{x}:\overline{n}|}, \quad A^{(m)}_{x:\overline{n}|} = A^{(m)}_{\overset{1}{x}:\overline{n}|} + {}_nE_x = \frac{i}{i^{(m)}} A_{\overset{1}{x}:\overline{n}|} + {}_nE_x$$

Let's derive $A^{(12)}_{\overset{1}{x}:\overline{1}|}$ under UDD (the uniform distribution of death between two integer ages). Under UDD, the number of deaths per month in Year 1 is $\dfrac{\ell_x - \ell_{x+1}}{12} = \dfrac{d_x}{12}$

$$A^{(12)}_{\overset{1}{x}:\overline{1}|} = \frac{d_x/12}{\ell_x}(v_j + v_j^2 + \ldots + v_j^{12}) = \frac{1}{12}\frac{d_x}{\ell_x} a_{\overline{12}|j} = q_x \frac{1}{12} a_{\overline{12}|j}$$

$$j = (1+i)^{1/12} - 1 \text{ is the monthly interest rate.}$$

$$A^{(1)}_{\overset{1}{x}:\overline{1}|} = A_{\overset{1}{x}:\overline{1}|} = \frac{d_x}{\ell_x} v_i = \frac{d_x}{\ell_x(1+i)}$$

$$\Rightarrow \frac{A^{(12)}_{\overset{1}{x}:\overline{1}|}}{A_{\overset{1}{x}:\overline{1}|}} = \frac{1}{12} a_{\overline{12}|j}(1+i) = \frac{1}{12}\frac{1-(1+j)^{-12}}{j}(1+i) = \frac{1}{12j}\left(1-(1+i)^{-1}\right)(1+i) = \frac{i}{12j} = \frac{i}{i^{(12)}}$$

Similarly,

$$\frac{A^{(12)}_{\overset{1}{x+1}:\overline{1}|}}{A_{\overset{1}{x+1}:\overline{1}|}} = \frac{A^{(12)}_{\overset{1}{x+2}:\overline{1}|}}{A_{\overset{1}{x+2}:\overline{1}|}} = \ldots = \frac{i}{i^{(12)}}$$

$$\Rightarrow \frac{A^{(12)}_{\overset{1}{x}:\overline{n}|}}{A_{\overset{1}{x}:\overline{n}|}} = \frac{A^{(12)}_{\overset{1}{x}:\overline{1}|} + {}_1E_x A^{(12)}_{\overset{1}{x+1}:\overline{1}|} + {}_2E_x A^{(12)}_{\overset{1}{x+2}:\overline{1}|} + \ldots + + {}_{n-1}E_x A^{(12)}_{\overset{1}{x+n-1}:\overline{1}|}}{A_{\overset{1}{x}:\overline{1}|} + {}_1E_x A_{\overset{1}{x+1}:\overline{1}|} + {}_2E_x A_{\overset{1}{x+2}:\overline{1}|} + \ldots + + {}_{n-1}E_x A_{\overset{1}{x+n-1}:\overline{1}|}} = \frac{i}{i^{(12)}}$$

Generally, under UDD, $A^{(m)}_{\overset{1}{x}:\overline{n}|} = \dfrac{i}{i^{(m)}} A_{\overset{1}{x}:\overline{n}|}$. Let $m \to \infty$: $i^{(m)} \to \delta$ and $\overline{A}_{\overset{1}{x}:\overline{n}|} = \dfrac{i}{\delta} A_{\overset{1}{x}:\overline{n}|}$ and $\overline{A}_x = \dfrac{i}{\delta} A_x$. Just remember

$$iA_{\overset{1}{x}:\overline{n}|} = i^{(m)} A^{(m)}_{\overset{1}{x}:\overline{n}|} = \delta \overline{A}_{\overset{1}{x}:\overline{n}|}$$

However, such a formula doesn't apply to an m-thly n-year endowment insurance of 1 on (x), for which the correct formula is

$$A^{(m)}_{x:\overline{n}|} = A^{(m)}_{\overset{1}{x}:\overline{n}|} + {}_nE_x = \frac{i}{i^{(m)}} A_{\overset{1}{x}:\overline{n}|} + {}_nE_x$$

9.3 UDD: claim acceleration approach

The average claim payment time in a 1-year m-thly term insurance is $0.5 + \dfrac{0.5}{m} = \dfrac{m+1}{2m} = 1 - \dfrac{m-1}{2m}$, which is $\dfrac{m-1}{2m}$ earlier than the end of Year 1. If we break down an n-year m-thly term insurance into an n consecutive 1-year m-thly term insurance contracts, we see that in each policy year the average claim time is $\dfrac{m-1}{2m}$ earlier than the end of the year. Hence we need to apply the factor of $(1+i)^{\frac{m-1}{2m}}$ to $A_{\overset{1}{x}:\overline{n}|}$.

$$A^{(m)}_{\overset{1}{x}:\overline{n}|} \approx (1+i)^{\frac{m-1}{2m}} A_{\overset{1}{x}:\overline{n}|}, \quad \overline{A}_{\overset{1}{x}:\overline{n}|} \approx (1+i)^{0.5} A_{\overset{1}{x}:\overline{n}|}$$

Under this approach, $A^{(12)}_{\overset{1}{x}:\overline{1}|} \approx (1+i)^{11/24} A_{\overset{1}{x}:\overline{1}|}$. Here's why. Under UDD, on average death occurs at $t = 0.5$ in Year 1. On average it takes the insurer half a month to process a claim. From receiving a death claim to investigating the claim to finally sending the check to the heir of the deceased takes on average half a month. Hence the death benefit 1 in $A^{(12)}_{\overset{1}{x}:\overline{1}|}$ is paid, on average, at $0.5 + 0.5/12 = 13/24 = 1 - 11/24$, which is $11/24$ earlier than when the death benefit of 1 is paid in $A_{\overset{1}{x}:\overline{1}|}$.

$$A^{(12)}_{\overset{1}{x}:\overline{1}|} \approx v^{1-11/24}\frac{d_x}{\ell_x} = v^{-11/24}\frac{d_x}{\ell_x} v = (1+i)^{11/24} A_{\overset{1}{x}:\overline{1}|}$$

Here's another way to derive the average payment time $\dfrac{13}{24}$ in $A^{(12)}_{\overset{1}{x}:\overline{1}|}$. Under UDD, (x) has an equal chance to die in each of the 12 months. Since the death benefit 1 is paid at the end of the month of death, the claim payment time is equally likely to be $1/12$, $2/12$, $3/12$, ..., and $12/12$. The average claim payment times is

$$\frac{1/12 + 2/12 + \ldots + 12/12}{12} = \frac{1+2+\ldots+12}{12^2} = \frac{0.5 \times 12 \times 13}{12^2} = \frac{13}{24} = 1 - \frac{11}{24}$$

Similarly,

$$A^{(12)}_{\overset{1}{x+1}:\overline{1}|} \approx (1+i)^{11/24} A_{\overset{1}{x+1}:\overline{1}|}$$

$$A^{(12)}_{\overset{1}{x+2}:\overline{1}|} \approx (1+i)^{11/24} A_{\overset{1}{x+2}:\overline{1}|}$$

$$\ldots$$

$$A^{(12)}_{\overset{1}{x}:\overline{n}|} = A^{(12)}_{\overset{1}{x}:\overline{1}|} + {}_1E_x A^{(12)}_{\overset{1}{x+1}:\overline{1}|} + {}_2E_x A^{(12)}_{\overset{1}{x+1}:\overline{1}|} + \ldots + {}_{n-1}E_x A^{(12)}_{\overset{1}{x+n-1}:\overline{1}|}$$

$$\approx (1+i)^{11/24}\left[A_{\overset{1}{x}:\overline{1}|} + {}_1E_x A_{\overset{1}{x+1}:\overline{1}|} + {}_2E_x A_{\overset{1}{x+1}:\overline{1}|} + \ldots + {}_{n-1}E_x A_{\overset{1}{x+n-1}:\overline{1}|}\right] = (1+i)^{11/24} A_{\overset{1}{x}:\overline{n}|}$$

Under UDD, the average claim payment time in $A^{(m)}_{\overset{1}{x}:\overline{1}|}$ is $\boxed{0.5 + \frac{0.5}{m} = \frac{m+1}{2m} = 1 - \frac{m-1}{2m}}$; $\boxed{A^{(m)}_{\overset{1}{x}:\overline{1}|} \approx (1+i)^{\frac{m-1}{2m}} A_{\overset{1}{x}:\overline{1}|}}$, $\boxed{A^{(m)}_{\overset{1}{x}:\overline{n}|} \approx (1+i)^{\frac{m-1}{2m}} A_{\overset{1}{x}:\overline{n}|}}$. Let $m \to \infty$: $\boxed{\overline{A}_{\overset{1}{x}:\overline{n}|} \approx (1+i)^{0.5} A_{\overset{1}{x}:\overline{n}|}}$. For an n-year endowment insurance, only the term insurance is subject to the claim acceleration approach: $\boxed{\overline{A}_{x:\overline{n}|} \approx (1+i)^{0.5} A_{\overset{1}{x}:\overline{n}|} + {}_nE_x}$.

9.4 EPV: m-thly n-year annuity due under UDD

For a life annuity due that pays $\frac{1}{m}$ at the beginning of each m-thly period for n years as long as (x) is alive, the PV random variable is

$$Y = \ddot{a}^{(m)}_{\overline{\min\left(K_x^{(m)}+\frac{1}{m},n\right)}|i} = \frac{1}{m}\ddot{a}_{\overline{\min(mK^{(m)}+1,mn)}|j}$$

$$j = (1+i)^{1/m} - 1$$

The EPV is

$$\ddot{a}^{(m)}_{x:\overline{n}|i} = \frac{1}{m}\ddot{a}_{x:\overline{mn}|j} = \frac{1}{m}\left(1 + v_j\,{}_{\frac{1}{m}}p_x + v_j^2\,{}_{\frac{2}{m}}p_x + \ldots + v_j^{mn-1}\,{}_{\frac{mn-1}{m}}p_x\right)$$

You should be able to calculate $\ddot{a}^{(12)}_{x:\overline{1}|i}$ using the first principle.

Now consider a life annuity due that pays $\frac{1}{12}$ at the beginning of each month for 1 year as long as (x) is alive. Its EPV is

$$\ddot{a}^{(12)}_{x:\overline{1}|} = \frac{1}{12}\ddot{a}_{x:\overline{12}|j} = \frac{1}{12}\left(1 + v_j\,{}_{\frac{1}{12}}p_x + v_j^2\,{}_{\frac{2}{12}}p_x + \ldots + v_j^{11}\,{}_{\frac{11}{12}}p_x\right)$$

$$j = (1+i)^{1/12} - 1 \text{ is the monthly interest rate}$$

Its PV random variable is

$$Y = \ddot{a}^{(12)}_{\overline{\min\left(K_x^{(12)}+\frac{1}{12},1\right)}|i} = \frac{1}{12}\ddot{a}_{\overline{\min(12K^{(12)}+1,12)}|j}$$

For example, if $K_x = 0.56$, then

$$K_x^{(12)} = \frac{\lfloor 0.56\times 12\rfloor}{12} = \frac{\lfloor 6.72\rfloor}{12} = \frac{6}{12}, \quad Y = \ddot{a}^{(12)}_{\overline{\frac{7}{12}}|i} = \frac{1}{12}\ddot{a}_{\overline{7}|j} = \frac{1}{12}\frac{1-v_j^7}{d_j}$$

Generally, for a life annuity due that pays $\frac{1}{m}$ at the beginning of each m-thly period for n years as long as (x) is alive, the PV random variable is

$$Y = \ddot{a}^{(m)}_{\overline{\min\left(K_x^{(m)}+\frac{1}{m},n\right)}|i} = \frac{1}{m}\ddot{a}_{\overline{\min(mK^{(m)}+1,mn)}|j}, \quad j = (1+i)^{1/m} - 1$$

$$\ddot{a}^{(m)}_{x:\overline{n}|i} = \frac{1}{m}\ddot{a}_{x:\overline{mn}|j} = \frac{1}{m}\left(1 + v_j\,{}_{\frac{1}{m}}p_x + v_j^2\,{}_{\frac{2}{m}}p_x + \ldots + v_j^{mn-1}\,{}_{\frac{mn-1}{m}}p_x\right)$$

Let's derive the formula for $\ddot{a}^{(12)}_{x:\overline{1}|i} = \frac{1}{12}\ddot{a}_{x:\overline{12}|j}$ under UDD. Under UDD,

$$f_x(t) = q_x, \quad {}_tp_x = 1 - tq_x, \quad 0 \le t \le 1$$

$$\ddot{a}_{x:\overline{12}|j} = 1 + v_j\left(1 - \frac{1}{12}q_x\right) + v_j^2\left(1 - \frac{2}{12}q_x\right) + \ldots + v_j^{11}\left(1 - \frac{11}{12}q_x\right)$$

$$= 1 + v_j + v_j^2 + \ldots + v_j^{11} - \frac{1}{12}q_x\left(v_j + 2v_j^2 + \ldots + 11v_j^{11}\right) = \ddot{a}_{\overline{12}|j} - \frac{1}{12}q_x(Ia)_{\overline{11}|j}$$

$$(Ia)_{\overline{n}|j} = \frac{\ddot{a}_{\overline{n}|} - nv_j^n}{j}, \quad \ddot{a}_{x:\overline{12}|j} = \ddot{a}_{\overline{12}|j} - \frac{1}{12}q_x\frac{\ddot{a}_{\overline{11}|} - v_j^{11}}{j}$$

Generally,

$$\ddot{a}^{(m)}_{x:\overline{1}|i} = \frac{1}{m}\ddot{a}_{x:\overline{m}|j} = \frac{1}{m}\left(\ddot{a}_{\overline{m}|j} - \frac{1}{m}q_x(Ia)_{\overline{m-1}|j}\right) = \frac{1}{m}\left(\ddot{a}_{\overline{m}|j} - \frac{1}{m}q_x\frac{\ddot{a}_{\overline{m-1}|} - (m-1)v_j^{m-1}}{j}\right)$$

$$\ddot{a}^{(m)}_{x:\overline{n}|i} = \ddot{a}^{(m)}_{x:\overline{1}|i} + {}_1E_x\ddot{a}^{(m)}_{x+1:\overline{1}|i} + {}_2E_x\ddot{a}^{(m)}_{x+2:\overline{1}|i} + \ldots + {}_{n-1}E_x\ddot{a}^{(m)}_{x+n-1:\overline{1}|i}$$
$$= \frac{1}{m}\left(\ddot{a}_{x:\overline{m}|j} + {}_1E_x\ddot{a}_{x+1:\overline{m}|j} + {}_2E_x\ddot{a}_{x+2:\overline{m}|j} + \ldots + {}_{n-1}E_x\ddot{a}_{x+n+1:\overline{m}|j}\right)$$

Example 9.4.1

- $q_x = 0.04$
- $i = 0.1$
- UDD holds.

Calculate $A_{\overset{1}{x}:\overline{1}|}$, $A^{(4)}_{\overset{1}{x}:\overline{1}|}$, $A^{(12)}_{\overset{1}{x}:\overline{1}|}$, $\overline{A}_{\overset{1}{x}:\overline{1}|}$.

Solution 9.4.1

Under UDD, for $0 \le t \le 1$, $f_x(t) = q_x$ and ${}_tp_x = 1 - tq_x$. $A^{(m)}_{\overset{1}{x}:\overline{1}|} = \frac{q_x}{m}a_{\overline{m}|j}$, $j = (1+i)^{1/m} - 1$. $A_{\overset{1}{x}:\overline{1}|} = vq_x = 0.04(1.1^{-1}) = 0.0363636$, $A^{(4)}_{\overset{1}{x}:\overline{1}|} = \frac{0.04}{4}a_{\overline{4}|j=1.1^{1/4}-1} = 0.0377002$, $A^{(12)}_{\overset{1}{x}:\overline{1}|} = \frac{0.04}{12}a_{\overline{4}|j=1.1^{1/12}-1} = 0.0380016$, $\overline{A}_{\overset{1}{x}:\overline{1}|} = \int_0^1 e^{-\delta t}f(t)dt = q_x\int_0^1 e^{-\delta t}dt = q_x\overline{a}_{\overline{1}|} = q_x\frac{1-v}{\delta} = 0.04\left(\frac{1-1.1^{-1}}{\ln 1.1}\right) = 0.0381529$. $A_{\overset{1}{x}:\overline{1}|} < A^{(4)}_{\overset{1}{x}:\overline{1}|} < A^{(12)}_{\overset{1}{x}:\overline{1}|} < \overline{A}_{\overset{1}{x}:\overline{1}|}$.

Example 9.4.2

- $q_x = 0.04$
- $i = 0.1$
- UDD holds.

Calculate $\ddot{a}_{x:\overline{1}|}$, $\ddot{a}^{(4)}_{x:\overline{1}|}$, $\ddot{a}^{(12)}_{x:\overline{1}|}$, $\overline{a}_{x:\overline{1}|}$.

Solution 9.4.2

$\ddot{a}_{x:\overline{1}|} = 1$, $\ddot{a}^{(4)}_{x:\overline{1}|} = \frac{1}{4}\left(\ddot{a}_{\overline{4}|j} - \frac{1}{4}q_x(Ia)_{\overline{3}|j}\right) = \frac{1}{4}\left(3.860929 - \frac{0.04}{4}5.676417\right) = 0.951041$, $j = 1.1^{1/4} - 1 = 2.41137\%$; $\ddot{a}^{(12)}_{x:\overline{1}|} = \frac{1}{12}\left(\ddot{a}_{\overline{12}|j} - \frac{1}{12}q_x(Ia)_{\overline{11}|j}\right) = \frac{1}{12}\left(11.491397 - \frac{0.04}{12}62.115208\right) = 0.940362$, $j = 1.1^{1/12} - 1 = 0.79741\%$; $\overline{a}_{x:\overline{1}|} = \int_0^1 {}_tp_xe^{-\delta t} = \int_0^1(1 - 0.04t)e^{-t\ln 1.1}dt = 0.9350500$. Notice that $\ddot{a}_{x:\overline{1}|} > \ddot{a}^{(4)}_{x:\overline{1}|} > \ddot{a}^{(12)}_{x:\overline{1}|} > \overline{a}_{x:\overline{1}|}$.

9.5 relationship: Y and Z for m-thly

UDD or not:

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}}, \quad Var[Y] = \frac{Var[Z]}{\left(d^{(m)}\right)^2} = \frac{{}^2A^{(m)}_{x:\overline{n}|} - \left(A^{(m)}_{x:\overline{n}|}\right)^2}{\left(d^{(m)}\right)^2}$$

Let Y represent the PV random variable of an n-year m-thly life annuity due of a total annual amount 1 on (x). Let Z represent the PV random variable of an n-year m-thly endowment insurance of 1 on (x). Then

$$Y = \ddot{a}^{(m)}_{\overline{\min\left(K_x^{(m)}+\frac{1}{m},n\right)}|}, \quad Z = v^{\min(K_x^{(m)}+\frac{1}{m},n)}$$

Using the FM formula: $\ddot{a}^{(m)}_{\overline{t}|} = \frac{1-v^t}{d^{(m)}}$, we have $Y = \frac{1-Z}{d^{(m)}}$.

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}}, \quad Var[Y] = \frac{Var[Z]}{\left(d^{(m)}\right)^2} = \frac{{}^2A^{(m)}_{x:\overline{n}|} - \left(A^{(m)}_{x:\overline{n}|}\right)^2}{\left(d^{(m)}\right)^2}$$

Example 9.5.1

- $q_x = 0.04$
- $i = 0.1$
- UDD holds.

Calculate $\ddot{a}^{(12)}_{x:\overline{1}|}$.

Solution 9.5.1

$A^{(12)}_{x:\overline{2}|} = A^{(12)}_{\overset{1}{x}:\overline{2}|} + {}_1E_x = \frac{i}{i^{(12)}}q_xv + p_xv = \frac{0.1}{12(1.1^{1/12}-1)}0.04(1.1^{-1}) + 0.96(1.1^{-1}) = 0.0380016 + 0.8727273 = 0.910729$, $\ddot{a}^{(12)}_{x:\overline{1}|} = \frac{1-0.910729}{12(1-1.1^{-1/12})} = 0.94036$

Example 9.5.2

- $q_x = 0.04$
- $q_{x+1} = 0.06$
- $q_{x+2} = 0.08$
- $i = 0.1$
- UDD holds.
- Y is the PV random variable of a 3-year life annuity due on (x) payable $\frac{1}{12}$ monthly
- Z is the PV random variable of a 3-year endowment insurance of 1 on (x) with the death benefit 1 payable at the end of the month of death

Calculate $Var[Y]$ and $Var[Z]$.

Solution 9.5.2

Think monthly! Make 2 new PV random variables: Y_{36} for a 36-month life annuity due of 1 on (x) and Z_{36} for a 36-month endowment insurance of 1 on (x). Now $E[Y_{36}] = \ddot{a}_{x:\overline{36}|j}$, $E[Z_{36}] = A_{x:\overline{36}|j}$, $j = 1.1^{1/12} - 1$, $Y_{36} = 12Y$, $Z_{36} = Y$.

$E[Z_{36}] = A^{\ 1}_{x:\overline{12}|j} + {}_{1|}A^{\ 1}_{x:\overline{12}|j} + {}_{2|}A^{\ 1}_{x:\overline{12}|j} + (1+i)^{-3}{}_3p_x$ $= \frac{a_{\overline{12}|j}}{12}[q_x + (1+i)^{-1}p_x q_{x+1} + (1+i)^{-2}p_x p_{x+1} q_{x+2}] = \frac{1-1.1^{-1}}{12(1.1^{1/12}-1)}\big(0.04+1.1^{-1}0.96(0.06)+1.1^{-2}0.96(0.94)0.08\big)+1.1^{-3}0.96(0.94)0.92 = 0.1444313 + 0.62374756 = 0.76817886$

To calculate $E[(Z_{36})^2]$, we just replace j with $j* = (1+j)^2 - 1$ and i with $i* = (1+i)^2 - 1$.

$E[(Z_{36})^2] = \frac{a_{\overline{12}|j*}}{12}[q_x + (1+i*)^{-1}p_x q_{x+1} + (1+i*)^{-2}p_x p_{x+1} q_{x+2}] = \frac{1-1.1^{-2}}{12(1.1^{2\times 1/12}-1)}\big(0.04 + 1.1^{-2}0.96(0.06)+1.1^{-4}0.96(0.94)0.08\big)+1.1^{-6}0.96(0.94)0.92 = 0.12366601 + 0.46863077 = 0.59229678$

$Var[Z_{36} = 0.59229678 - 0.76817886^2 = 0.0021980$, monthly $d = 1 - 1.1^{-1/12}$, $Var[Y_{36}] = \dfrac{0.0021980}{d^2} = 74.869491$, $Var[Y_{36}] = 74.869491/12^2 = 0.243893$. By the way, $E[Y_{36}] = \dfrac{1-0.76817886}{d} = 29.303431$, and $E[Y] = 29.303431/12 = 2.4419530$

9.6 relationship: EPV of m-thly annuity due, immediate

$$a^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{m} + \frac{1}{m}{}_nE_x = \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{m}(1 - {}_nE_x)$$

Example 9.6.1

- Mortality: Illustrative Life Table.
- $i = 0.06$
- $\ddot{a}^{(12)}_{45:\overline{10}|} = 7.42945$

Calculate $a^{(12)}_{45:\overline{10}|}$.

Solution 9.6.1

$a^{(12)}_{45:\overline{10}|} = \ddot{a}^{(12)}_{45:\overline{10}|} - \frac{1}{12}(1 - {}_{10}E_{45}) = 7.42945 - \frac{1}{12}(1 - 0.526515) = 7.39000$

9.7 Euler-Maclaurin formula

Euler-Maclaurin formula is an improved version of the trapezoidal rule. The trapezoidal rule approximation is:

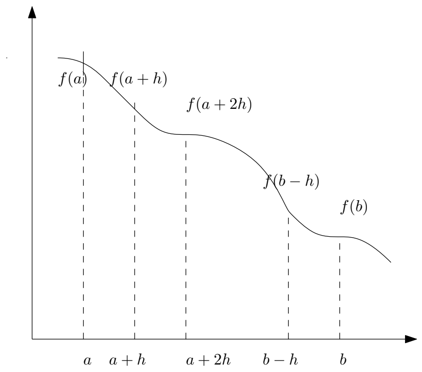
$$\int_a^b f(x)ds \approx \frac{h}{2}[f(a)+f(a+h)] + \frac{h}{2}[f(a+h)+f(a+2h)] + \ldots + \frac{h}{2}[f(a+(n-1)h)+f(b)]$$
$$= h\Big[f(a)+f(a+h)+\ldots+f(b)\Big] - \frac{h}{2}\Big[f(a)+f(b)\Big]$$
$$= h\Big[f(a)+f(a+h)+\ldots+f(b-h)\Big] - \frac{h}{2}\Big[f(a)-f(b)\Big]$$

Euler-Maclaurin formula is:

$$\int_a^b f(x)dx \approx h\Big[f(a)+f(a+h)+\ldots+f(b)\Big] - \frac{h}{2}\Big[f(a)+f(b)\Big] + \frac{h^2}{12}\Big[f^1(a)-f^1(b)\Big]$$
$$= h\Big[f(a)+f(a+h)+\ldots+f(b-h)\Big] - \frac{h}{2}\Big[f(a)-f(b)\Big] + \frac{h^2}{12}\Big[f^1(a)-f^1(b)\Big]$$

The textbook uses the Euler-Maclaurin formula to derive the Woolhouse's formula so you need to know a little bit of the Euler-Maclaurin formula. I'll give you a cliff notes version of the Euler-Maclaurin formula. This formula was independently discovered by both Euler and Maclaurin. It was the very formula by which Euler guessed $\sum_{n=0}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, which he later proved.

Suppose we need to estimate $\int_a^b f(x)dx$, the area under the curve $f(x)$ bounded by $x = a$ and $x = b$. Before the Euler-Maclaurin formula, we would use the trapezoidal rule to approximate this area.



Divide the interval $[a, b]$ into m intervals each of length $h = \frac{b-a}{m}$.
$\int_a^b f(x)dx$ is roughly the sum of the areas of m trapezoids (R is the error term):

$$\int_a^b f(x)ds =$$

$$\frac{h}{2}[f(a)+f(a+h)]+\frac{h}{2}[f(a+h)+f(a+2h)]+\ldots+\frac{h}{2}[f(a+(n-1)h)+f(b)]+R$$

$$= h[f(a) + f(a + h) + \ldots + f(b)] - \frac{h}{2}[f(a) + f(b)] + R$$

The Euler-Maclaurin formula specifies the error term:

$$R = \frac{b_2}{2!}[f^1(b) - f^1(a)]h^2 + \frac{b_3}{3!}[f^2(b) - f^2(a)]h^3 + \ldots + \frac{b_{k+1}}{(k+1)!}[f^k(b) - f^k(a)]h^{k+1} + R_2$$

b_i is a Bernoulli's number (to be explained shortly), f^i is i-th derivative of $f(x)$, k is any positive integer as long as $f(x)$ has a continuous k-th derivative, and R_2 is an error term.
The Bernoulli's number b_k for $k = 0, 1, 2, \ldots$ is defined by the equation

$$\frac{x}{e^x - 1} = b_0 + b_1 x + \frac{b_2}{2!}x^2 + \frac{b_3}{3!}x^3 + \ldots + \frac{b_k}{k!}x^k + \ldots = \sum_{k=0}^{\infty} \frac{b_k}{k!}x^k$$

$$b_k = \frac{d^n}{dx^n}\left(\frac{x}{e^x - 1}\right)\Bigg|_{x=0}$$

$$b_0 = \lim_{x\to 0} \frac{x}{e^x - 1} = \lim_{x\to 0} \frac{1}{e^x} = 1, \quad b_1 = \lim_{x\to 0} \frac{d}{dx}\frac{x}{e^x - 1} = \lim_{x\to 0} \frac{-1 + e^x - xe^x}{(e^x - 1)^2} = \lim_{x\to 0} \frac{-x}{2(e^x - 1)} = \lim_{x\to 0} \frac{-1}{2e^x} = -\frac{1}{2}$$

Here's another way to find b_k. Consider the reverse function $\frac{e^x - 1}{x}$. Use Taylor expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\frac{e^x - 1}{x} = 1 + u, \quad u = \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \ldots$$

Now the original function is

$$\frac{x}{e^x - 1} = \frac{1}{1 + u} = 1 + (-u) + (-u)^2 + (-u)^3 + \ldots$$

To find b_0 to b_4, we'll omit the x^5 terms and above.

$$\frac{1}{1+u} = 1 - \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120}\right) + \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24}\right)^2 - \left(\frac{x}{2} + \frac{x^2}{6}\right)^2 + \left(\frac{x}{2}\right)^4 + o(x^5)$$

$$= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \ldots = \sum_{k=0}^{\infty} \frac{b_k}{k!}x^k$$

$$\frac{b_0}{0!} = 1, \quad \frac{b_1}{1!} = -\frac{1}{2}, \quad \frac{b_2}{2!} = \frac{1}{12}, \quad \frac{b_3}{3!} = 0, \quad \frac{b_4}{4!} = -\frac{1}{720}$$

Similarly, we can find that

$$\frac{b_5}{5!} = 0, \quad \frac{b_6}{6!} = \frac{1}{30240}$$

By the way, the odd Bernoulli's numbers are all zero except $b_1 = -\frac{1}{2}$. That is, $b_{2k-1} = 0$ for $k \geq 2$. To summarize, under the Euler-Maclaurin formula:

$$\int_a^b f(x)dx = h[f(a) + f(a + h) + \ldots + f(b)] - \frac{h}{2}[f(a) + f(b)] + R$$

$$R = \frac{h^2}{12}[f^1(a) - f^1(b)] - \frac{h^4}{720}[f^3(a) - f^3(b)] + \frac{h^6}{30240}[f^a(b) - f^5(b)] + \ldots + R_2 = \sum_{i=2}^{k} \frac{b_i}{i!}h^i[f^{i-1}(a) - f^{i-1}(b)] + R_2$$

.

Ignoring the third and higher derivatives:

$$\int_a^b f(x)dx \approx h\Big[f(a) + f(a + h) + \ldots + f(b)\Big] - \frac{h}{2}\Big[f(a) + f(b)\Big] + \frac{h^2}{12}\Big[f^1(a) - f^1(b)\Big]$$

$$\approx h\Big[f(a) + f(a + h) + \ldots + f(b - h)\Big] - \frac{h}{2}\Big[f(a) - f(b)\Big] + \frac{h^2}{12}\Big[f^1(a) - f^1(b)\Big]$$

9.8 Woolhouse's formula

Because the 3-term Woolhouse approximation is too time-consuming, most likely, all you need to do in the exam is to use the following 2-term Woolhouse's formula:

$$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x)$$

The 2-term Woolhouse's formula is exactly the trapezoidal rule.

If you are short on time, just memorize the 2-term Woolhouse's formula and move on. Don't bother understanding the 3-term Woolhouse formula or the Euler-Maclaurin formula.

Let's apply the Euler-Maclaurin formula to the integral $\overline{a}_{x:\overline{n}|} = \int_0^n g(t)dt = \int_0^n {}_tE_x dt$.

$$g(t) = {}_tE_x = e^{-\delta t}{}_tp_x, \quad g(0) = 1, \quad g(n) = {}_nE_x$$

$$g^1(t) = -{}_tp_x\delta e^{-\delta t} - e^{-\delta t}{}_tp_x\mu_{x+t} = -{}_tE_x(\delta + \mu_{x+t})$$

$$g^1(0) = -(\delta + \mu_x), \quad g^1(n) = -{}_nE_x(\delta + \mu_{x+n})$$

$$\overline{a}_{x:\overline{n}|} \approx h\Big[g(0) + g(h) + g(2h) + \ldots + g(n-h)\Big] - \frac{h}{2}(1 - {}_nE_x) - \frac{h^2}{12}\Big[(\delta + \mu_x) - {}_nE_x(\delta + \mu_{x+n})\Big]$$

Set $h = \dfrac{1}{m}$ where m is a positive integer.

$$h\Big[g(0) + g(h) + g(2h) + \ldots + g(n-h)\Big] = \frac{1}{m}\Big[1 + {}_{\frac{1}{m}}E_x + {}_{\frac{2}{m}}E_x + \ldots + {}_{\frac{nm-1}{m}}E_x\Big] = \ddot{a}_{x:\overline{n}|}^{(m)}$$

$$\boxed{\overline{a}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{2m}(1 - {}_nE_x) - \frac{1}{12m^2}\Big[(\delta + \mu_x) - {}_nE_x(\delta + \mu_{x+n})\Big]}$$

The above formula should work for any positive integer m. Set $m = 1$ and we get a 3-term Woolhouse approximation:

$$\boxed{\overline{a}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{1}{2}(1 - {}_nE_x) - \frac{1}{12}\Big[(\delta + \mu_x) - {}_nE_x(\delta + \mu_{x+n})\Big] \approx \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{2m}(1 - {}_nE_x) - \frac{1}{12m^2}\Big[(\delta + \mu_x) - {}_nE_x(\delta + \mu_{x+n})\Big]}$$

3-term Woolhouse approximation:

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|} - \left(\frac{1}{2} - \frac{1}{2m}\right)(1 - {}_nE_x) - \left(\frac{1}{12} - \frac{1}{12m^2}\right)\Big[(\delta + \mu_x) - {}_nE_x(\delta + \mu_{x+n})\Big]$$

Set $n = \infty$. Then $g(\infty) = g^1(\infty) = 0$:

$$\boxed{\overline{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\delta + \mu_x) \approx \ddot{a}_x^{(m)} - \frac{1}{2m} - \frac{1}{12m^2}(\delta + \mu_x)}$$

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \left(\frac{1}{2} - \frac{1}{2m}\right) - \left(\frac{1}{12} - \frac{1}{12m^2}\right)(\delta + \mu_x)$$

2-term Woolhouse approximation (e.g. trapezoidal rule):

$$\boxed{\overline{a}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{1 - {}_nE_x}{2} \approx \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1 - {}_nE_x}{2m}, \quad \ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x)}$$

$$\overline{a}_x \approx \ddot{a}_x - \frac{1}{2} \approx \ddot{a}_x^{(m)} - \frac{1}{2m}$$

Example 9.8.1

(Exam MLC: Spring 2015 Q7) You are given:

(i) $A_{35} = 0.188$

(ii) $A_{65} = 0.498$

(iii) ${}_{30}p_{35} = 0.883$

(iv) $i = 0.04$

Calculate $1000\ddot{a}_{35:\overline{30}|}^{(2)}$ using the two-term Woolhouse approximation.

Solution 9.8.1

$\overline{a}_{35:\overline{30}|} \approx \ddot{a}_{35:\overline{30}|} - \dfrac{1 - {}_{30}E_{35}}{2} \approx \ddot{a}_{35:\overline{30}|}^{(2)} - \dfrac{1 - {}_{30}E_{35}}{2 \times 30}$,

$\ddot{a}_{35:\overline{30}|} = \ddot{a}_{35} - \ddot{a}_{65\,30}E_{35} = \dfrac{1 - 0.188}{0.04/1.04} - \dfrac{1 - 0.498}{0.04/1.04}(0.883)1.04^{-30} = 17.558653$

$17.558653 - \dfrac{1 - (0.883)1.04^{-30}}{2} \approx \ddot{a}_{35:\overline{30}|}^{(30)} - \dfrac{1 - (0.883)1.04^{-30}}{2 \times 30}$, $\ddot{a}_{35:\overline{30}|}^{(2)} \approx 17.206905$, $1000(17.206905) = 17,207$

9.9 EPV of m-thly whole life annuity due: alpha and beta under UDD

Instead of the 2-term Woolhouse's formula,

$$\ddot{a}^{(m)}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x)$$

now

$$\ddot{a}^{(m)}_{x:\overline{n}|} \approx \alpha(m)\ddot{a}_{x:\overline{n}|} - \beta(m)\frac{m-1}{2m}(1 - {}_nE_x), \quad \alpha(m) = \frac{id}{i^{(m)}d^{(m)}}, \quad \beta(m) = \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}$$

UDD or not:

$$\ddot{a}^{(m)}_x = \frac{1 - A^{(m)}_x}{d^{(m)}}, \quad A^{(m)}_x = 1 - d\ddot{a}^{(m)}_x$$

UDD:

$$A^{(m)}_x = \frac{i}{i^{(m)}}A_x$$

$$\ddot{a}^{(m)}_x = \frac{1 - \frac{i}{i^{(m)}}(1 - d\ddot{a}_x)}{d^{(m)}} = \frac{1}{i^{(m)}d^{(m)}}\left[id\ddot{a}_x - (i - i^{(m)})\right] = \alpha(m)\ddot{a}_x - \beta(m)$$

$$\alpha(m) = \frac{id}{i^{(m)}d^{(m)}}, \quad \beta(m) = \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}$$

If $m \to \infty$, then $i^{(m)} = d^{(m)} = \delta$ and $\bar{a}_x = \frac{1}{\delta^2}\left[id\ddot{a}_x - (i - \delta)\right]$.

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_x - {}_nE_x\ddot{a}^{(m)}_{x+n} = \left[\alpha(m)\ddot{a}_x - \beta(m)\right] - {}_nE_x\left[\alpha(m)\ddot{a}_{x+n} - \beta(m)\right] = \alpha(m)\ddot{a}_{x:\overline{n}|} - \beta(m)(1 - {}_nE_x)$$

We can verify that $\alpha(m) \approx 1$ and $\beta(m) \approx \frac{1}{2} - \frac{1}{2m}$. Use Taylor expansion:

$$i = e^\delta - 1 \approx \delta + \frac{1}{2}\delta^2, \quad d = 1 - v = 1 - e^{-\delta} \approx \delta - \frac{1}{2}\delta^2$$

$$i^{(m)} = m((1+i)^{1/m} - 1) = m(e^{\delta/m} - 1) \approx m(\frac{\delta}{m} + \frac{\delta}{2m^2}) = \delta + \frac{\delta}{2m}$$

$$d^{(m)} = m(1 - (1+i)^{-1/m}) = m(1 - e^{-\delta/m}) \approx m(\frac{\delta}{m} - \frac{\delta}{2m^2}) = \delta - \frac{\delta}{2m}$$

$$id \approx \delta^2, \quad i^{(m)}d^{(m)} \approx \delta^2, \quad i - i^{(m)} \approx \delta^2\left(\frac{1}{2} - \frac{1}{2m}\right)$$

$$\Rightarrow \quad \alpha(m) = \frac{id}{i^{(m)}d^{(m)}} \approx 1, \quad \beta(m) = \frac{i - i^{(m)}}{i^{(m)}d^{(m)}} \approx \frac{1}{2} - \frac{1}{2m}$$

Simplified approximation:

$$\ddot{a}^{(m)}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \left(\frac{1}{2} - \frac{1}{2m}\right)(1 - {}_nE_x)$$

Let $m \to \infty$, then

$$i^{(m)} = d^{(m)} = \delta, \quad \alpha(m) = \frac{id}{\delta^2}, \quad \beta(m) = \frac{i - \delta}{\delta^2}$$

Under UDD,

$$\bar{a}_{x:\overline{n}|} = \frac{id}{\delta^2}\ddot{a}_{x:\overline{n}|} - \frac{i - \delta}{\delta^2}(1 - {}_nE_x)$$

$$\bar{a}_x = \frac{id}{\delta^2}\ddot{a}_x - \frac{i - \delta}{\delta^2}$$

Example 9.9.1

For $S(x) = 1 - \dfrac{x}{100}$ where $0 \le x \le 100$ and $i = 0.10$, calculate $\ddot{a}^{(12)}_{40:\overline{3}|}$ under the following methodologies:

(i) Exact

(ii) UDD (uniform distribution of deaths between integral ages)

(iii) $W2$: 2-term Woolhouse approximation

(iv) $W3$: 3-term Woolhouse approximation with exact force of mortality

(v) $W3*$: 3-term Woolhouse approximation with approximate force of mortality

Solution 9.9.1

Notice the morality is De Moivre's law. Arbitrarily set $\ell_0 = 100$.

$$\ell_x = \ell_0 S(x) = 100 - x, \quad {}_tp_{40} = \frac{\ell_{40+t}}{\ell_{40}} = \frac{60-t}{60}, \quad f_{40}(t) = -\frac{d}{dt}{}_tp_{40} = \frac{1}{60}$$

$$\mu_{40+t} = \frac{f_{40}(t)}{{}_tp_{40}} = \frac{1}{60-t}, \quad \ddot{a}_{40:\overline{3}|} = 1 + \frac{59}{60}\left(1.1^{-1}\right) + \frac{58}{60}\left(1.1^{-2}\right) = 2.692837$$

EXACT. Generally, it's hard to manually calculate $\ddot{a}^{(m)}_{x:\overline{n}|}$. However,it's easy to calculate under the De Moivre's law. Let $j = 1.1^{1/12} - 1$ represent the monthly interest rate. $v_j = 1/(1+j)$.

$$12\ddot{a}^{(12)}_{40:\overline{3}|} = \frac{60}{60} + \frac{60 - 1/12}{60}v_j + \frac{60 - 2/12}{60}v_j^2 + \ldots + \frac{60 - \frac{3\times 12 - 1}{12}}{60}v_j^{35} = \ddot{a}_{\overline{36}|j} - \frac{1}{60\times 12}(Ia)_{\overline{35}|j}$$

Remember that ${}_tp_{40} = \dfrac{60-t}{60}$ and that there are 36 monthly payments.

$$\ddot{a}_{\overline{36}|j} = \frac{1 - v_j^{36}}{d_j} = \frac{1 - 1.1^{-3}}{1 - 1.1^{-1/12}} = 31.435144$$

$$(Ia)_{\overline{35}|j} = \frac{\ddot{a}_{\overline{35}|} - 35v_j^{35}}{j} = \frac{\dfrac{1 - 1.1^{-35/12}}{1 - 1.1^{-1/12}} - 35\times 1.1^{-35/12}}{1.1^{1/12} - 1} = 523.20769$$

$$12\ddot{a}^{(12)}_{40:\overline{3}|} = 31.435144 - \frac{523.20769}{60\times 12} = 30.708467, \quad \ddot{a}^{(12)}_{40:\overline{3}|} = \frac{30.708467}{12} = 2.5590389$$

If ℓ_x is proportional to $\omega - x$ (De Moivre's law), then for $n = 1, , \ldots, \omega - x$ and $j = (1+i)^{1/m} - 1$,

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1}{m}\left(\frac{\omega - x}{\omega - x} + \frac{\omega - x - \frac{1}{m}}{\omega - x}v_j + \frac{\omega - x - \frac{2}{m}}{\omega - x}v_j^2 + \ldots + \frac{\omega - x - \frac{nm-1}{m}}{\omega - x}v_j^{nm-1}\right)$$

$$\boxed{\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1}{m}\left(\ddot{a}_{\overline{mn}|j} - \frac{1}{m(\omega - x)}(Ia)_{\overline{mn-1}|j}\right)}$$

UDD.

$$i = 0.1, d = 0.1/1.1, i^{(12)} = 12(1.1^{1/12} - 1) = 0.095690, d^{(12)} = 12(1 - 1.1^{-1/12}) = 0.094933$$

$$\alpha(12) = \frac{id}{i^{(12)}d^{(12)}} = 1.0007520, \beta(12) = \frac{i - i^{(12)}}{i^{(12)}d^{(12)}} = 0.4744912$$

$$\ddot{a}^{(12)}_{40:\overline{3}|} = \alpha(12)\ddot{a}_{40:\overline{3}|} - \beta(12)(1 - {}_3E_{40}) = 1.0007520(2.692837) - 0.4744912\left(1 - \frac{57}{60}(1.1^{-3})\right) = 2.559039$$

$$W2: \quad \ddot{a}^{(12)}_{40:\overline{3}|} \approx \ddot{a}_{40:\overline{3}|} - \frac{1}{2}\left(1 - \frac{1}{12}\right)(1 - {}_3E_{40})$$

$$= 2.692837 - \frac{1}{2}\left(1 - \frac{1}{12}\right)\left(1 - \frac{57}{60}(1.1^{-3})\right) = 2.561639$$

$$W3 = W2 - \frac{1}{12}\left(1 - \frac{1}{12^2}\right)\left(\delta + \mu_{40} - {}_3E_{40}(\delta + \mu_{43})\right)$$

$$= W2 - \frac{1}{12}\left(1 - \frac{1}{12^2}\right)\left(\ln 1.1 + \frac{1}{60} - \frac{57}{60}1.1^{-3}(\ln 1.1 + \frac{1}{57})\right) = W2 - 0.0026008 = 2.561639 - 0.0026008 = 2.55904$$

To calculate $W3*$, repeat the $W3$ calculation but use the approximate μ_{40} and μ_{43}. The approximation is

$$\mu_x \approx -0.5(\ln p_{x-1} + \ln p_x) = -0.5\ln(p_{x-1}p_x) = -0.5(\ln\frac{\ell_{x+1}}{\ell_{x-1}}) = -0.5\ln {}_2p_{x-1}$$

$$\mu_{40} \approx -0.5\ln\frac{\ell_{41}}{\ell_{39}} = -0.5\ln\frac{100-41}{100-39} = 0.016668, \quad \mu_{43} \approx -0.5\ln\frac{100-44}{100-42} = 0.017546$$

$$W3* = W2 - \frac{1}{12}\left(1-\frac{1}{12^2}\right)\left(\delta+\mu_{40}-{}_3E_{40}(\delta+\mu_{43})\right)$$
$$= W2 - \frac{1}{12}\left(1-\frac{1}{12^2}\right)\left(\ln 1.1+0.016668-\frac{57}{60}1.1^{-3}(\ln 1.1+0.017546)\right)$$
$$= 2.561639-0.0026008 = 2.561639-0.0026008 = 2.55904$$

Example 9.9.2

For $S(x) = 1-\frac{x}{100}$ where $0 \le x \le 100$ and $i = 0.10$, calculate $\ddot{a}_{40}^{(12)}$ using Exact, UDD, $W2$, $W3$, and $W3*$.

Solution 9.9.2

EXACT METHOD 1.

$$12\ddot{a}_{40}^{(12)} = 12\ddot{a}_{40:\overline{60}|}^{(12)} = \ddot{a}_{\overline{12\times 60}|j} - \frac{1}{60\times 12}(Ia)_{\overline{12\times 60-1}|j}$$
$$\ddot{a}_{\overline{12\times 60}|j} = \frac{1-1.1^{-60}}{1-1.1^{-1/12}} = 125.99022$$
$$(Ia)_{\overline{12\times 60-1}|j} = \frac{\frac{1-1.1^{-(12\times 60-1)/12}}{1-1.1^{-1/12}} - (12\times 60-1)\times 1.1^{-(12\times 60-1)/12}}{1.1^{1/12}-1} = 15500.942$$
$$12\ddot{a}_{40}^{(12)} = 125.99022 - \frac{15500.942}{60\times 12} = 104.46113, \quad \ddot{a}_{40}^{(12)} = \frac{104.46113}{12} = 8.7050942$$

EXACT METHOD 2. We'll use the formula $\ddot{a}_x^{(m)} = \frac{1-A_x^{(m)}}{d^{(m)}}$. First, we'll develop a formula $A_x^{(m)} = \frac{1}{\omega-x}a_{\overline{\omega-x}|}^{(m)}$. De Moivre's law satisfies UDD.

$$i^{(m)}A_x^{(m)} = \delta\overline{A}_x$$
$$\overline{A}_x = \int_0^{\omega-x} e^{-\delta t}f_x(t)dt = \int_0^{\omega-x} e^{-\delta t}\frac{1}{\omega-x}dt = \frac{1}{\omega-x}\frac{1-v^{-(\omega-x)}}{\delta}$$
$$A_x^{(m)} = \frac{1}{\omega-x}\frac{1-v^{-(\omega-x)}}{i^{(m)}} = \frac{1}{\omega-x}a_{\overline{\omega-x}|}^{(m)}$$
$$A_{40}^{(12)} = \frac{1}{60}a_{\overline{60}|}^{(12)} = \frac{1}{60}\frac{1-1.1^{-60}}{12(1.1^{1/12}-1)} = 0.17360208$$
$$\ddot{a}_{40}^{(12)} = \frac{1-A_x^{(12)}}{d^{(12)}} = \frac{1-0.17360208}{12(1-1.1^{-1/12})} = 8.7050943$$

UDD.

$$A_{40} = \frac{1}{60}a_{\overline{60}|} = \frac{1}{60}\frac{1-1.1^{-60}}{0.1} = 0.16611929$$
$$\ddot{a}_{40} = \frac{1-A_{40}}{d} = \frac{1-0.16611929}{0.1/1.1} = 9.1726878$$
$$\ddot{a}_{40}^{(12)} = \alpha(12)\ddot{a}_{40} - \beta(12) = 1.0007520(9.1726878) - 0.4744912 = 8.7050945$$

$$W2: \quad \ddot{a}_{40}^{(12)} \approx \ddot{a}_{40} - \frac{1}{2}\left(1-\frac{1}{12}\right) = 9.1726878 - \frac{1}{2}\left(1-\frac{1}{12}\right) = 8.7143545$$
$$W3 = W2 - \frac{1}{12}\left(1-\frac{1}{12^2}\right)(\delta+\mu_{40}) = 8.7143545 - \frac{1}{12}\left(1-\frac{1}{12^2}\right)\left(\ln 1.1+\frac{1}{60}\right) = 8.7050879$$

$W3*$: repeat the $W3$ calculation but use

$$\mu_{40} \approx -0.5\ln\frac{\ell_{41}}{\ell_{39}} = -0.5\ln\frac{100-41}{100-39} = 0.016668$$
$$W3* = W2 - \frac{1}{12}\left(1-\frac{1}{12^2}\right)(\delta+\mu_{40}) = 8.7143545 - \frac{1}{12}\left(1-\frac{1}{12^2}\right)(\ln 1.1+0.016668) = 8.7050878$$

Example 9.9.3

For $S(x) = 1 - \frac{x}{100}$ where $0 \le x \le 100$ and $i = 0.10$, calculate $\bar{a}_{40}$ using Exact, UDD, $W2$, $W3$, and $W3*$.

Solution 9.9.3

$$EXACT \quad \bar{a}_{40} = \frac{1-\bar{A}_{40}}{\delta} = \frac{1-\frac{1}{60}\bar{a}_{\overline{60}|}}{\delta} = \frac{1-\frac{1}{60}\frac{1-1.1^{-60}}{\ln 1.1}}{\ln 1.1} = 8.6633628$$

$$\ddot{a}_{40} = \frac{1-A_{40}}{d} = \frac{1-\frac{1}{60}a_{\overline{60}|}}{d} = \frac{1-\frac{1}{60}\frac{1-1.1^{-60}}{0.1}}{0.1/1.1} = 9.1726878$$

$$UDD: \quad \bar{a}_{40} = \alpha(\infty)\ddot{a}_{40} - \beta(\infty) = 1.000757232 \times 9.1726878 - 0.516270862 = 8.6633628$$

$$W2: \quad \bar{a}_{40} \approx \ddot{a}_{40} - 0.5 = 9.1726878 - 0.5 = 8.6726878$$

$$W3 = W2 - \frac{1}{12}(\delta + \mu_{40}) = 8.6726878 - \frac{1}{12}\left(\ln 1.1 + \frac{1}{60}\right) = 8.6633564$$

$W3*$: repeat the $W3$ calculation but use

$$\mu_{40} \approx -0.5 \ln \frac{\ell_{41}}{\ell_{39}} = -0.5 \ln \frac{100-41}{100-39} = 0.016668$$

$$W3* = 8.6726878 - \frac{1}{12}(\ln 1.1 + 0.016668) = 8.6633563$$

9.10 Check your knowledge

Homework 9.10.1

Use the Illustrative Life Table and $i = 8\%$. Calculate $\ddot{a}^{(4)}_{60:\overline{3}|}$ using UDD, $W2$, and $W3*$.

Homework Solution 9.10.1

★★★★☆ Difficulty

$$\ddot{a}_{60:\overline{3}|} = \frac{\ell_{60} + \ell_{61}v + \ell_{62}v^2}{\ell_{60}} = 2.746033$$

$$i = 0.08, \quad d = 0.08/1.08$$

$$i^{(4)} = 4(1.08^{1/4} - 1) = 0.077706, \quad d^{(4)} = 4(1 - 1.08^{-1/4}) = 0.076225$$

$$UDD: \quad \alpha(4) = \frac{id}{i^{(4)}d^{(4)}} = 1.000463, \quad \beta(4) = \frac{i - i^{(4)}}{i^{(4)}d^{(4)}} = 0.387260$$

$$\ddot{a}^{(4)}_{60:\overline{3}|} = \alpha(4)\ddot{a}_{60:\overline{3}|} - \beta(4)(1 - {}_3E_{60}) = 1.000463(2.746033) - 0.387260(1 - 0.758524) = 2.65379$$

$$W2: \quad \ddot{a}^{(4)}_{60:\overline{3}|} \approx \ddot{a}_{60:\overline{3}|} - \frac{1}{2}\left(1 - \frac{1}{4}\right)(1 - {}_3E_{60}) = 2.746033 - \frac{1}{2}\left(1-\frac{1}{4}\right)(1 - 0.758524) = 2.65548$$

$$W3*: \quad \mu_{60} \approx -0.5 \ln \frac{\ell_{61}}{\ell_{59}} = 0.013277201, \quad \mu_{63} \approx -0.5 \ln \frac{\ell_{64}}{\ell_{62}} = 0.017279969$$

$$W3* = W2 - \frac{1}{12}\left(1 - \frac{1}{4^2}\right)\left(\delta + \mu_{60} - {}_3E_{60}(\delta + \mu_{63})\right)$$

$$= 2.65548 - \frac{1}{12}\left(1 - \frac{1}{4^2}\right)\left(\ln 1.08 + 0.013277201 - 0.758524(\ln 1.08 + 0.017279969)\right)$$

$$= 2.65401$$

Homework 9.10.2

(i) Mortality: Standard Ultimate Survival Table (in Appendix D of AMLRC textbook)

(ii) $i = 0.05$

(iii) $\ddot{a}_{50:\overline{20}|} = 12.842791$

Calculate $\ddot{a}^{(12)}_{50:\overline{20}|}$ using UDD, $W2$, $W3$, and $W3*$.

Homework Solution 9.10.2

★★★★☆ Difficulty

$i = 0.05$, $i^{(12)} = 0.048889$, $d^{(12)} = 0.048691$, $\alpha(12) = 1.000197$, $\beta(12) = 0.466508$,

	UDD	$W2$	$W3$	$W3*$
$\ddot{a}^{(12)}_{50:\overline{20}\|}$	12.54127	12.54407	12.54162	12.54162

$\mu_x = A + Bc^x$, where $A = 0.00022$, $B = 0.0000027$, $c = 1.124$, $\mu_{50} = 0.0011526$, $\mu_{70} = 0.0098806$

For $W3*$, the estimated force of mortalities are $\mu_{50} \approx -\ln {}_2p_{49} = -\ln \dfrac{\ell_{51}}{\ell_{49}} = 0.0011547$, $\mu_{70} \approx -0.5 \ln {}_2p_{69} = -0.5 \ln \dfrac{\ell_{71}}{\ell_{69}} = 0.0099026$

Homework 9.10.3

(i) Mortality: Standard Select Ultimate Survival Table (in Appendix D of AMLRC textbook)

(ii) $i = 0.05$

(iii) $\ddot{a}_{50:\overline{20}|} = 12.845595$

Calculate $\ddot{a}^{(12)}_{50:\overline{20}|}$ using UDD, $W2$, and W.

Homework Solution 9.10.3

★★★★☆ Difficulty

$i = 0.05$, $i^{(12)} = 0.048889$, $d^{(12)} = 0.048691$, $\alpha(12) = 1.000197$, $\beta(12) = 0.466508$,

	UDD	$W2$	$W3$	$W3*$
$\ddot{a}^{(12)}_{50:\overline{20}|}$	12.54411	12.54691	12.54449	12.54448

$\mu_{[x]+s} = 0.9^{2-s}\mu_{x+s}$ for $s = 0, 1, 2$, $\mu_{x+s} = A + Bc^{x+s}$, where $A = 0.00022$, $B = 0.0000027$, $c = 1.124$, $\mu_{[50]} = 0.9^2 \times 0.0011526 = 0.0009336$, $\mu_{70} = 0.0098806$. For $W3*$, the estimated force of mortalities are $\mu_{[50]} \approx -\ln p_{[50]} = -\ln \dfrac{\ell_{[50]+1}}{\ell_{[50]}} = 0.0010338$, $\mu_{70} \approx -0.5 \ln {}_2p_{69} = -0.5 \ln \dfrac{\ell_{71}}{\ell_{69}} = 0.0099026$

Homework 9.10.4

(i) Mortality: Illustrative Life Table

(ii) $i = 0.06$

Calculate ${}_{20|}\ddot{a}^{(12)}_{45}$ using UDD, $W2$, and $W3*$.

Homework Solution 9.10.4

★★★★☆ Difficulty

${}_{20|}\ddot{a}^{(12)}_{45} = {}_{20}E_{45}\ddot{a}^{(12)}_{65}$, ${}_{20}E_{45} = 0.256341$

UDD: $\ddot{a}^{(12)}_{65} = \alpha(12)\ddot{a}_{65} - \beta(12) = 1.000281 \times 9.89693 - 0.468120) = 9.43159$, $0.256341(9.43159) = 2.417703$

$W2$: $\ddot{a}^{(12)}_{65} = \ddot{a}_{65} - \dfrac{1}{2}\left(1 - \dfrac{1}{12}\right) = 9.43859$, $0.256341(9.43859) = 2.419498$

$W3*$: $9.43859 - \dfrac{1}{12}\left(1 - \dfrac{1}{12^2}\right)(\ln 1.06 + \mu_{65}) = 9.432065$, where $\mu_{65} \approx -0.5 \ln \dfrac{\ell_{66}}{64} = -0.5 \ln \dfrac{7373338}{7683979} = 0.0206335$, $0.256341(9.432065) = 2.4178250$

Homework 9.10.5

(i) Mortality: Illustrative Life Table

(ii) $i = 0.06$

Calculate $a^{(4)}_{30:\overline{20}|}$ using UDD, $W2$, and $W3*$.

Homework Solution 9.10.5

★★★★☆ Difficulty

$a^{(4)}_{30:\overline{20}|} = \ddot{a}^{(4)}_{30:\overline{20}|} - \dfrac{1}{4}(1 - {}_{20}E_{30}) = \ddot{a}^{(4)}_{30:\overline{20}|} - 0.17657$

$\ddot{a}_{30:\overline{20}|} = \ddot{a}_{30} - {}_{20}E_{30}\ddot{a}_{30} = 11.95913$, ${}_{20}E_{30} = 0.29374$

	UDD	$W2$	$W3*$
$\ddot{a}^{(4)}_{30:\overline{20}|}$	11.69093	11.69428	11.69108
$a^{(4)}_{30:\overline{20}|}$	11.51436	11.51771	11.51451

Homework 9.10.6

For a semi-continuous 20-year decreasing insurance on (40), you are given:

(i) Death benefit: paid at the moment of death.

(ii) Death benefit amount: 200,000 for each of the first 10 years and 100,000 for each of the next 10 years.

(iii) Annual premium: P for each of the first 10 years and $0.5P$ for each of the next 10 years.

(iv) EPV of continuous death benefits is calculated using the claim acceleration approach

Premium basis:

- Mortality: Illustrative Life Table
- $i = 6\%$.
- Expense: zero
- Selective actuarial values: $A^{\,1}_{40:\overline{20}|} = 0.060132$, $A^{\,1}_{40:\overline{10}|} = 0.027667$, $\ddot{a}_{40:\overline{20}|} = 11.76126$, $\ddot{a}_{40:\overline{10}|} = 7.69664$

Calculate P.

Homework Solution 9.10.6

★★★★☆ Difficulty
Under the claim acceleration method, the continuous death claim payment throughout the year is modeled by one claim payment at mid year.

$$0.5P\ddot{a}_{40:\overline{20|}} + 0.5P\ddot{a}_{40:\overline{10|}} = 100,000\overline{A}^{\,1}_{40:\overline{20|}} + 100,000\overline{A}^{\,1}_{40:\overline{10|}}$$

$$P = 100000\frac{\overline{A}^{\,1}_{40:\overline{20|}} + \overline{A}^{\,1}_{40:\overline{10|}}}{0.5(\ddot{a}_{40:\overline{20|}} + \ddot{a}_{40:\overline{10|}})} = 100000 \times 1.06^{0.5} \times \frac{A^{\,1}_{40:\overline{20|}} + A^{\,1}_{40:\overline{10|}}}{0.5(\ddot{a}_{40:\overline{20|}} + \ddot{a}_{40:\overline{10|}})}$$

$$= 100000(1.06^{0.5}) \times \frac{0.060132 + 0.027667}{0.5(11.76126 + 7.69664)} = 929.13$$

Homework 9.10.7

An insurer issues an annuity to a life 60. The annuity is payable monthly in advance and is guaranteed for the first 10 years and for the whole life thereafter. If the annual payment is 2,000, calculate the EPV of this annuity. Basis:

- Mortality: Illustrative Life Table
- $i = 6\%$

Homework Solution 9.10.7

★★★★☆ Difficulty

$$EPV = 2000\left(\ddot{a}^{(12)}_{\overline{10|}} + {}_{10}E_{60}\ddot{a}^{(12)}_{70}\right) = 2000(7.597117 + 0.45120 \times 8.1109) = 22514$$

$$\ddot{a}^{(12)}_{\overline{10|}} = \frac{1 - v^{10}}{d^{(12)}} = \frac{1 - 1.06^{-10}}{0.058128} = 7.597117, \quad \ddot{a}^{(12)}_{70} = \ddot{a}_{70} - \frac{11}{24} = 8.5693 - \frac{11}{24} = 8.1109$$

Homework 9.10.8

Explain $\ddot{a}^{(12)}_{50:50:\overline{20|}}$ in English and calculate its value using the Illustrative Life Table and 6% interest.

Homework Solution 9.10.8

★★★★★ Difficulty
$\ddot{a}^{(12)}_{50:50}$ is the expected present value of an annuity on two lives (50) and (50). The payment is 1 per year for a maximum of 20 years payable monthly in advance while both lives are alive.

$$\ddot{a}^{(12)}_{50:50:\overline{20|}} = \ddot{a}^{(12)}_{50:50} - {}_{20}E_{50:50}\,\ddot{a}^{(12)}_{70:70}$$

$$\ddot{a}^{(12)}_{50:50} = \ddot{a}_{50:50} - \frac{11}{24} = 11.6513 - \frac{11}{24} = 11.1930, \quad \ddot{a}^{(12)}_{70:70} = \ddot{a}_{70:70} - \frac{11}{24} = 6.5247 - \frac{11}{24} = 6.0664$$

$${}_{20}E_{50:50} = v^{20}{}_{20}p_{50}\,{}_{20}p_{50} = {}_{20}E_{50}\,{}_{20}p_{50} = 0.23047 \times 0.73916$$

$$\ddot{a}^{(12)}_{50:50:\overline{20|}} = 11.1930 - 0.23047 \times 0.73916 \times 6.0664 = 10.1596$$

Homework 9.10.9

For a reversionary annuity on a man age 60 and his wife age 60, you are given:

(i) Payments: 10,000 per year payable monthly for life to the wife after the death of the husband. The first payment occurs one month after the death of the husband. Assume that the husband dies at the end of a month.

(ii) Premiums: level monthly premiums payable immediately after issue while both the husband and the wife are alive.

(iii) Expenses: 4% of each premium payment and 2% of each annuity payment

Calculate the monthly premium under the equivalence principle using the Illustrative Life Table and 6% interest.

Homework Solution 9.10.9

★★★★★ Difficulty

$$0.96 \times 12P\ddot{a}^{(12)}_{60:60} = 1.02 \times 10,000a^{(12)}_{60|60} = 1.02 \times 10,000(a^{(12)}_{60} - a^{(12)}_{60:60})$$

$$\ddot{a}^{(12)}_{60:60} = \ddot{a}_{60:60} - \frac{11}{24} = 9.1911 - \frac{11}{24} = 8.7328$$

$$a^{(12)}_{60} - a^{(12)}_{60:60} = \ddot{a}^{(12)}_{60} - \ddot{a}^{(12)}_{60:60} = \ddot{a}_{60} - \frac{11}{24} - \left(\ddot{a}_{60:60} - \frac{11}{24}\right) = \ddot{a}_{60} - \ddot{a}_{60:60} = 11.1454 - 9.1911 = 1.9543$$

$$0.96 \times 12P \times 8.7328 = 1.02 \times 10000(1.9543)$$

$$P = 198.15$$

Homework 9.10.10

For a special fully discrete whole life insurance of 100,000 on (50). You are given:

(i) Mortality: Illustrative Life Table

(ii) $i = 6\%$

(iii) Premium: level monthly

Calculate the net level monthly premium assuming

- level monthly payable for at most 10 years
- level monthly payable for life

Homework Solution 9.10.10

★★★★☆ Difficulty

$$\ddot{a}^{(12)}_{50:\overline{10}|} = \ddot{a}_{50:\overline{10}|} - \frac{11}{24}\left(1 - {}_{10}E_{50}\right) = 7.57371 - \frac{11}{24}(1 - 0.510806) = 7.349496$$

$$12P\ddot{a}^{(12)}_{50:\overline{10}|} = 100000A_{50}, \quad 12 \times 7.349496P = 100000 \times 0.24905, \quad P = 282.39$$

$$\ddot{a}^{(12)}_{50} = \ddot{a}_{50} - \frac{11}{24} = 13.26683 - \frac{11}{24} = 12.80850$$

$$12 \times 12.80850P = 100000 \times 0.24905, \quad P = 162.03$$

Homework 9.10.11

For a special fully discrete whole life insurance on (50), you are given:

- Death benefit: 100,000 at the end of the month of death
- Premium: level monthly premium
- Mortality: Illustrative Life Table
- $i = 6\%$
- Initial expense: 10% of the total of the first year premiums incurred at issue
- Commission: 4% of each monthly premium including premiums in Year 1

(a) Calculate the monthly premium using the equivalence principle

(b) Write down the loss at issue random variable for this policy

(c) The insurer issues 10,000 identical policies to independent lives age 50. Calculate the monthly premium such that the probability of a positive total loss is 1%.

Homework Solution 9.10.11

★★★★★ Difficulty

$$(a) \quad 12P\ddot{a}^{(12)}_{50} = 100,000A^{(12)}_{50} + 0.1 \times 12P + 0.04 \times 12P\ddot{a}^{(12)}_{50}, \quad P = \frac{1}{12} \times \frac{100,000A^{(12)}_{50}}{0.96\ddot{a}^{(12)}_{50} - 0.1}$$

We assume UDD.

$$100,000A^{12}_{50} = 100,000 \times \frac{i}{i^{(12)}}A_{50} = 100000 \times \frac{0.06}{12 \times (1.06^{1/12} - 1)} \times 0.24905 = 25582.682$$

$$\ddot{a}^{(12)}_{50} = \ddot{a}_{50} - \frac{11}{24} = 13.26683 - \frac{11}{24} = 12.808495$$

$$P = \frac{1}{12} \times \frac{25582.682}{0.96 \times 12.808495 - 0.1} = 174.80$$

$(b), (c)$ The total loss at issue is

$$S = L_1 + L_2 + \ldots + L_N, \quad N = 10,000, \quad L_i's \quad \textit{are iid (independent identically distributed)}$$

S is approximately normal.

$$E(S) = NE(L), \quad Var(S) = NVar(L)$$

$$L = 100,000v^{K^{(12)}_{50} + \frac{1}{12}} + 0.1 \times 12P - 0.96 \times 12P\ddot{a}^{(12)}_{\overline{K^{(12)}_{50} + \frac{1}{12}}|}$$

$$= 100,000v^{K^{(12)}_{50} + \frac{1}{12}} + 0.1 \times 12P - 0.96 \times 12P \times \frac{1 - v^{K^{(12)}_{50} + \frac{1}{12}}}{d^{(12)}}$$

$$= \left(100,000 + \frac{0.96 \times 12}{d^{(12)}}P\right)v^{K^{(12)}_{50} + \frac{1}{12}} + 12P\left(0.1 - \frac{0.96}{d^{(12)}}\right)$$

$$= \left(100000 + \frac{0.96 \times 12}{0.058128} P\right) v^{K_{50}^{(12)} + \frac{1}{12}} + 12P \left(0.1 - \frac{0.96}{0.058128}\right)$$

$$= (100000 + 198.18332P) v^{K_{50}^{(12)} + \frac{1}{12}} - 196.98332P$$

$$d^{(12)} = 12 \left(1 - 1.06^{-1/12}\right) = 5.81277\%$$

$$E(L) = (100000 + 198.18332P) A_{60}^{(12)} - 196.98332P$$

$$= (100000 + 198.18332P) 0.255824 - 196.98332P = 25582.4 - 146.28327P$$

$$Var(L) = (100000 + 198.18332P)^2 \left({}^2A_{50}^{(12)} - \left(A_{50}^{(12)}\right)^2\right)$$

$${}^2A_{50}^{(12)} = A_{50|j=1.06^2-1}^{(12)} = \frac{j}{j^{(12)}} A_{50|j} = \frac{1.06^2 - 1}{12 \times (1.06^{2/12} - 1)} \times 0.094756 = \frac{0.1236}{0.11710553} \times 0.094756 = 0.100011$$

$${}^2A_{50}^{(12)} - \left(A_{50}^{(12)}\right)^2 = 0.100011 - 0.255824^2 = 0.034565$$

$$E(S) = 10000(25582.4 - 146.28327P), \quad Var(S) = 10000(100000 + 198.18332P)^2 \times 0.034565$$

$$\sigma_S = 100(100000 + 198.18332P)\sqrt{0.034565}$$

$$Pr\left(S < 0\right) = \Phi\left(\frac{0 - 10000(25582.4 - 146.28327P)}{100(100000 + 198.18332P)\sqrt{0.034565}}\right) = 0.99$$

$$\frac{0 - 10000(25582.4 - 146.28327P)}{100(100000 + 198.18332P)\sqrt{0.034565}} = \Phi^{-1}(0.99) = 2.3263$$

$$P = 178.88$$

Homework 9.10.12

For a special fully discrete 10-year endowment insurance on (60), you are given:

- Death benefit: 100,000 at the end of the month of death
- Premium: level monthly premium
- Mortality: Illustrative Life Table
- $i = 6\%$
- Commission: 5% of each monthly premium including premiums in Year 1
- Selective actuarial value: ${}^2A_{50:\overline{10}|}^{(12)} = 0.357714$

(a) Calculate the monthly premium using the equivalence principle

(b) The insurer issues 10,000 identical policies to independent lives age 60. Calculate the monthly premium such that the probability of a positive total loss is 10%.

Homework Solution 9.10.12

★★★★★ Difficulty

$$12(0.95)P\ddot{a}_{60:\overline{10}|}^{(12)} = 100,000A_{60:\overline{10}|}^{(12)}$$

We assume UDD.

$$A_{60:\overline{10}|}^{(12)} = A_{\stackrel{1}{60}:\overline{10}|} \frac{i}{i^{(12)}} + {}_{10}E_{60}$$

$$A_{\stackrel{1}{60}:\overline{10}|} = A_{60} - {}_{10}E_{60}A_{70} = 0.36913 - 0.451196 \times 0.514948 = 0.13679$$

$$A_{60:\overline{10}|}^{(12)} = \frac{0.06}{12\left(1.06^{1/12} - 1\right)} 0.13679 + 0.451196 = 0.591707$$

$$\ddot{a}_{60:\overline{10}|}^{(12)} = \ddot{a}_{60:\overline{10}|} - \frac{11}{24}\left(1 - {}_{10}E_{60}\right)$$

$$\ddot{a}_{60:\overline{10}|} = \ddot{a}_{60} - \ddot{a}_{70\,10}E_{60} = 11.14535 - 8.56925 \times 0.451196 = 7.27894$$

$$\ddot{a}_{60:\overline{10}|}^{(12)} = 7.27894 - \frac{11}{24}(1 - 0.451196) = 7.027405$$

$$12(0.95)P\ddot{a}^{(12)}_{60:\overline{10}|} = \frac{100,000A^{(12)}_{60:\overline{10}|}}{12(0.95)\ddot{a}^{(12)}_{60:\overline{10}|}} = \frac{100000 \times 0.591707}{12(0.95)7.027405} = 738.60$$

The total loss at issue is

$$S = L_1 + L_2 + \ldots + L_N, \quad N = 10,000, \quad L_i's \quad \textit{are iid (independent identically distributed)}$$

S is approximately normal.

$$E(S) = NE(L), \quad Var(S) = NVar(L)$$

$$L = 100,000v^{K^{(12)}_{60}+\frac{1}{12}} - 0.95 \times 12P\ddot{a}^{(12)}_{\overline{K^{(12)}_{60}+\frac{1}{12}}|}$$

$$= 100,000v^{K^{(12)}_{60}+\frac{1}{12}} - 0.95 \times 12P \times \frac{1 - v^{K^{(12)}_{60}+\frac{1}{12}}}{d^{(12)}}$$

$$= \left(100,000 + \frac{0.95 \times 12}{d^{(12)}}P\right) v^{K^{(12)}_{60}+\frac{1}{12}} - P\frac{12 \times 0.95}{d^{(12)}}$$

$$= (100000 + 196.11891P)\, v^{K^{(12)}_{60}+\frac{1}{12}} - 196.11891P$$

$$d^{(12)} = 12\left(1 - 1.06^{-1/12}\right) = 5.81277\%$$

$$E(L) = (100000 + 196.11891P)0.591707 - 196.11891P = 59170.7 - 80.073978P$$

$$Var(L) = (100000 + 196.11891P)^2 \left({}^2A^{(12)}_{60:\overline{10}|} - \left(A^{(12)}_{60:\overline{10}|}\right)^2\right)$$

$$= (100000 + 196.11891P)^2 \left(0.357714 - 0.591707^2\right)$$

$$\sigma_L = (100000 + 196.11891P)\sqrt{0.357714 - 0.591707^2} = (100000 + 196.11891P)0.087160$$

$$E(S) = 10000(59170.7 - 80.073978P), \quad \sigma_S = 100(100000 + 196.11891P)0.087160$$

$$Pr\,(S < 0) = \Phi\left(\frac{0 - 10000(59170.7 - 80.073978P)}{100(100000 + 196.11891P)0.087160}\right) = 0.95$$

$$\frac{0 - 10000(59170.7 - 80.073978P)}{100(100000 + 196.11891P)0.087160} = \Phi^{-1}(0.9) = 1.2816$$

$$P = 742.37$$

Chapter 27

Reversionary bonus: loss-at-issue, policy value

27.1 concept

Prerequisite. Before reading this chapter, you'll need to read Chapter 26 "Reversionary bonus and geometrically increasing benefit: find EPV" and Chapter 14 "Loss-at-issue random variable, equivalence principle, net premium, gross premium."

After you understand how reversionary bonuses work and how to set up the equivalence equation, you'll need to identify all the cash flows: premiums, original benefits, the accrued reversionary bonuses, and various expenses. Make sure you understand the difference between bonuses being added at the beginning of the year (BOY) and the bonuses being at the end of the year (EOY). Since it's very easy to miss a cash flow especially an expense component, you'll want to painstakingly keep track of each cash flow. Ask yourself: "Have I used all the information given to me?" If not, chances are that you have dropped something.

27.2 illustrative problems

Example 27.2.1

For a special fully discrete with-profit whole insurance on (50), you are given:

- initial insurance: 100,000

The insurer prices the product on the following basis:

(i) Mortality: the Illustrative Life Table

(ii) $i = 0.06$

(iii) Bonuses: simple bonus rate 3% per year. The bonus is added at the end of the year.

(iv) initial expenses - per policy: 120 plus 50% of Year 1 premium

(v) renewal expenses - per policy: 15 plus 2% of the renewal premiums

(vi) per 1,000 expense: 0.1 in Year 1 and 0.01 in renewal years

(vii) claims expenses: 100 per claim

(viii) selective actuarial values: $(IA)_{50} = 4.99329$, $(I\ddot{a})_{50} = 146.104583$

What you need to do:

(a) Write down the expression for the gross premium loss-at-issue random variable

(b) Calculate the gross premium using the equivalence principle

Solution 27.2.1

PV

- total insurance

$$100,000(1 + 0.03K_{50})v^{K_{50}+1}$$
$$= 100,000\Big(0.97 + 0.03(K_{50} + 1)\Big)v^{K_{50}+1}$$

- claim expense $100v^{K_{50}+1}$
- premium expense $(0.02\ddot{a}_{\overline{K_{50}+1|}} + 0.48)P$
- per policy expense $15\ddot{a}_{\overline{K_{50}+1|}} + 105$
- per 1,000 expense

$$= 100\Big(0.97 + 0.03(K_{50} + 1)\Big)0.01\ddot{a}_{\overline{K_{50}+1|}} + 100 \times 0.09$$
$$= \Big(0.97 + 0.03(K_{50} + 1)\Big)\ddot{a}_{\overline{K_{50}+1|}} + 9$$

- total premiums $P\ddot{a}_{\overline{K_{50}+1|}}$

EPV

- death benefit $97,000A_{50} + 3000(IA)_{50}$
- claim expense $100A_{50}$
- premium expense $(0.02\ddot{a}_{50} + 0.48)P$
- per policy expense $15\ddot{a}_{50} + 105$
- per 1,000 expense $0.97\ddot{a}_{50} + 0.03(I\ddot{a})_{50} + 9$
- total premiums $P\ddot{a}_{50}$

$$L_g^0 = \Big(97,100 + 3,000(K_{50}+1)\Big)v^{K_{50}+1} + \Big(15.97 + 0.03(K_{50}+1)\Big)\ddot{a}_{\overline{K_{50}+1}|} + 114 - P(0.98\ddot{a}_{0.98\overline{K_{50}+1}|} - 0.48)$$

$$P(0.98\ddot{a}_{50} - 0.48) = 97,100A_{50} + 3000(IA)_{50} + 15.97\ddot{a}_{50} + 0.03(I\ddot{a})_{50} + 114$$

$$P(0.98 \times 13.26683 - 0.48) = 97100 \times 0.24905 + 3000 \times 4.99676 + 15.97 \times 13.26683 + 0.03 \times 146.104583 + 114$$

$$P = \frac{39,503.04}{12.521491} = 3154.8$$

Example 27.2.2

For a special fully discrete 20-year with-profit endowment insurance policy on (45) with the initial sum insured of 100,000, you are given the following gross premium basis:

(i) Mortality: the Illustrative Life Table

(ii) $i = 0.06$

(iii) Premium: level gross monthly premium P calculated using the equivalence principle

(iv) The insurer declares a simple reversionary bonus 2% per year. The bonus is added at the end of the year

(v) initial expenses: $150 + 0.6(12P)$, that is, a fixed cost 150 plus 60% of Year 1 total gross premiums

(vi) renewal expenses: 3% of the second and subsequent monthly premiums

(vii) claims expenses: 100 on death; 50 on maturity

Selective actuarial values:

- $A^{\,1}_{45:\overline{20}|} = 0.088464$, ${}_{20}E_{45} = 0.256341$, $\ddot{a}_{45:\overline{20}|} = 11.57510$, $(IA)^{\,1}_{45:\overline{20}|} = 0.97740$
- $A^{\,1}_{45:\overline{10}|} = 0.040540$, ${}_{10}E_{45} = 0.526515$, $\ddot{a}_{45:\overline{10}|} = 7.64869$, $(IA)^{\,1}_{45:\overline{10}|} = 0.22851$, ${}_{10}p_{45} = 0.942908$
- $A^{\,1}_{55:\overline{10}|} = 7.457346$, ${}_{10}E_{55} = 0.486864$, $\ddot{a}_{55:\overline{10}|} = 0.091022$

What you need to do:

(a) Use the Woolhouse formula, calculate $\ddot{a}^{(12)}_{45:\overline{20}|}$.

(b) Verify that $P = 377.71$

(c) 10 years after issue and immediately before the then premium due and immediately after the bonus is awarded, the insurer decides to pay a terminal bonus to each surviving policyholder. The terminal bonus is equal to 90% of the retrospective gross premium policy value. The gross premium policy value is calculated on the same basis as the gross premium basis. Assume that insurer declares the annual 2% reversionary bonus at the end of each year throughout the contract term consistent with the gross premium assumption. Calculate the terminal bonus at $t = 10$.

(d) Explain whether the terminal bonus at $t = 10$ will be larger, the same, or smaller if the terminal bonus is equal to 90% of the prospective gross premium policy value, instead of 90% of the retrospective gross premium policy value, with the prospective policy value, the retrospective policy value, and the gross premium on the same basis.

(e) Calculate the prospective policy value at $t = 10$ under the same basis for which the retrospective gross premium value is calculated.

Solution 27.2.2

EPV of each item:

- monthly gross premiums: $12P\ddot{a}^{(12)}_{45:\overline{20}|}$
- initial death benefits: $100,000A^{\,1}_{45:\overline{20}|}$
- annual bonuses: $2,000(IA)^{\,1}_{45:\overline{20}|} - 2,000A^{\,1}_{45:\overline{20}|}$
- maturity benefit: $100,000(1 + 0.02 \times 20)\,{}_{20}E_{45}$
- death claim expense: $100A^{\,1}_{45:\overline{20}|}$
- maturity claim expense: $50\,{}_{20}E_{45}$
- initial expense: $150 + 0.6(12P)$
- renewal expense: $(0.03)12P\ddot{a}^{(12)}_{45:\overline{20}|} - 0.03P$

$(a),(b)$ $$\Big((0.97)12\ddot{a}^{(12)}_{45:\overline{20}|} - 0.6(12) + 0.03\Big)P = (100000 - 2000 + 100)A^{\,1}_{45:\overline{20}|} + 2000(IA)^{\,1}_{45:\overline{20}|} + (140000 + 50)\,{}_{20}E_{45} + 150$$

$$\ddot{a}^{(12)}_{45:\overline{20}|} = \ddot{a}_{45:\overline{20}|} - \frac{11}{24}(1 - {}_{20}E_{45}) = 11.57510 - \frac{11}{24}(1 - 0.256341) = 11.23426$$

$$\Big((0.97)12(11.23426) - 0.6(12) + 0.03\Big)P = (100000 - 2000 + 100)0.088464 + 2000 \times 0.97740 + (140000 + 50)0.256341 + 150$$

$$123.59679P = 46683.535, \quad P = 377.71$$

$$(c)\quad {}_{10}V^{Retro} = \frac{1.06^{10}}{{}_{10}p_{45}}\left((0.97\times 12\ddot{a}^{(12)}_{45:\overline{10}|} + 0.03 - 0.6\times 12)P - (100000 - 2000 + 100)A^{\;1}_{45:\overline{10}|} - 2000(IA)^{\;1}_{45:\overline{10}|} - 150\right)$$

$$= \frac{1.06^{10}}{0.942908}\left((0.97\times 12\times 7.43168 + 0.03 - 0.6\times 12)377.71 - (98100)0.040540 - 2000\times 0.22851 - 150\right) = 48206.70$$

$$\ddot{a}^{(12)}_{45:\overline{10}|} = \ddot{a}_{45:\overline{10}|} - \frac{11}{24}(1 - {}_{10}E_{45}) = 7.64869 - \frac{11}{24}(1 - 0.526515) = 7.43168$$

$$0.9(48206.70) = 43386.03$$

(*d*) The terminal bonus will be the same. The prospective and the retrospective gross premium policy values will be the same because (1) the gross premium is based on the equivalence principle, and (2) the policy value basis is the same as the gross premium basis.

(*e*). Ten years after issue, for a policy still in force, we see a 10-year endowment policy. The policy has accrued 20,000 bonuses and will declare a 2,000 bonus each year. EPVs are:

- monthly gross premiums: $12P\ddot{a}^{(12)}_{55:\overline{10}|}$
- initial benefits: $100,000A^{\;1}_{55:\overline{10}|}$
- accrued bonuses: $20,000A^{\;1}_{55:\overline{10}|}$
- future bonuses: $2,000(IA)^{\;1}_{55:\overline{10}|} - 2,000A^{\;1}_{55:\overline{10}|}$
- maturity benefit: $100,000(1 + 0.02\times 20)\,{}_{10}E_{55}$
- death claim expense: $100A^{\;1}_{55:\overline{10}|}$
- maturity claim expense: $50\,{}_{10}E_{55}$
- renewal expense: $(0.03)12P\ddot{a}^{(12)}_{55:\overline{10}|}$

$${}_{10}V = (100,000 + 20,000 + 100 - 2,000)A^{\;1}_{55:\overline{10}|} + 2,000(IA)^{\;1}_{55:\overline{10}|} + 140050\,{}_{10}E_{55} - (0.97)12P\ddot{a}^{(12)}_{55:\overline{10}|}$$

$$\ddot{a}^{(12)}_{55:\overline{10}|} = \ddot{a}_{55:\overline{10}|} - \frac{11}{24}(1 - {}_{10}E_{55}) = 7.45735 - \frac{11}{24}(1 - 0.486864) = 7.22216$$

$$(IA)^{\;1}_{45:\overline{20}|} = (IA)^{\;1}_{45:\overline{10}|} + {}_{10}E_{45}\left((IA)^{\;1}_{55:\overline{10}|} + 10A^{\;1}_{55:\overline{10}|}\right)$$

$$0.97740 = 0.22851 + 0.526515\left((IA)^{\;1}_{55:\overline{10}|} + 10\times 0.091022\right),\quad (IA)^{\;1}_{55:\overline{10}|} = 0.51212$$

$${}_{10}V = 118100\times 0.091022 + 2000\times 0.51212 + 140050\times 0.486864 - (0.97)12\times 377.71\times 7.22216 = 48206.70$$

GENERAL FORMULA: for $m \le n$:

$$(IA)^{\;1}_{x:\overline{n}|} = (IA)^{\;1}_{x:\overline{m}|} + {}_{m}E_{x}\left((IA)^{\;\;1}_{x+m:\overline{n-m}|} + mA^{\;\;1}_{x+m:\overline{n-m}|}\right)$$

Example 27.2.3

For a special semi-continuous 20-pay with-profit whole insurance on (50), you are given:

- the initial insurance is 100,000
- death benefit is paid at the moment of death

The insurer prices the product on the following basis:

(i) Mortality: the Standard Select Survival Model

(ii) $i = 0.05$

(iii) Bonuses: simple bonus rate 2% per year. The bonus is added at the end of the year.

(iv) initial expenses: 150 plus 60% of Year 1 premium

(v) renewal expenses: 3% of the renewal premiums

(vi) claims expenses: 100 on death

(vii) EPV of continuous term insurance valuation method: claim acceleration approach

(viii) selective actuarial values: $A_{[50]} = 0.18913$, $(IA)_{[50]} = 5.82662$, $\ddot{a}_{[50]:\overline{20}|} = 12.84560$

What you need to do:

(a) Write down the expression for the gross premium loss-at-issue random variable

(b) Calculate the gross premium using the equivalence principle

(c) Ten years after issue, bonuses totaling 20,000 have been declared. Calculate the prospective net premium reserve for the policy at $t = 10$ using the Illustrative Life Table and $i = 0.06$.

(d) In (c), if the retrospective reserving method is used, will the reserve be different from the prospective reserve?

(e) Calculate the net premium reserve at $t = 1, 2, 3$ respectively using both the retrospective method and the prospective method. Explain which reserving method, the retrospective or the prospective, is more prudent.

Solution 27.2.3

$$(a) \quad L_g^0 = 150 + 0.57P + (100,100 + 2,000K_{[50]})v^{T_{[50]}} - 0.97P\ddot{a}_{\overline{\min(1+K_{[50]},20)}|}$$

$$\text{claim acceleration approach} \Rightarrow T_{[x]} = K_{[x]} + 0.5 = (K_{[x]} + 1) - 0.5$$

$$L_g^0 = 150 + 0.57P + (100,100 + 2,000K_{[50]})v^{K_{[50]}+1}(1+i)^{0.5} - 0.97P\ddot{a}_{\overline{\min(1+K_{[50]},20)}|}$$

$$= 150 + 0.57P + \Big(98,100 + 2,000(K_{[50]} + 1)\Big)v^{K_{[50]}+1}(1+i)^{0.5} - 0.97P\ddot{a}_{\overline{\min(1+K_{[50]},20)}|}$$

One common mistake is to forget that Standard Select Survival Model has a two year selection period and write

$$L_g^0 = 150 + 0.57P + (100,100 + 2,000K_{50})v^{T_{50}} - 0.97P\ddot{a}_{\overline{\min(1+K_{50},20)}|}$$

$$(b) \quad E(L_g^0) = 0: \quad 150 + 0.57P + 98,100A_{[50]}(1+i)^{0.5} + 2,000(1+i)^{0.5}(IA)_{[50]} - 0.97P\ddot{a}_{[50]:\overline{20}|} = 0$$

$$P = \frac{150 + 98100A_{[50]}(1+i)^{0.5} + 2000(1+i)^{0.5}(IA)_{[50]}}{0.97\ddot{a}_{[50]:\overline{20}|} - 0.57}$$

$$= \frac{150 + 98100 \times 0.18913(1.05^{0.5}) + 2000(1.05^{0.5})5.82662}{0.97 \times 12.84560 - 0.57} = \frac{31102.47}{11.890227} = 2,615.80$$

(c) Net premium reserves consider only the guaranteed benefits. Since no bonus is guaranteed at issue, the net premium ignores any bonuses. However, the total accrued bonuses 20,000 during the first 10 years are guaranteed and need to be included in ${}_{10}V^n$.

$$\pi = 100000 \times \frac{\overline{A}_{50}}{\ddot{a}_{50:\overline{20}|}} = 100000 \times \frac{1.06^{0.5} \times 0.24905}{11.29184} = 2,270.76$$

The Illustrative Life Table is an ultimate table. There's no need to write [50].

METHOD 1

$${}_{10}V^n = 120000\overline{A}_{60} - \pi\ddot{a}_{60:\overline{10}|} = 120000 \times 1.06^{0.5}A_{60} - \pi\ddot{a}_{60:\overline{10}|} = 120000 \times 1.06^{0.5} \times 0.36913 - 2270.76 \times 7.27894 = 29076$$

The net premium reserve at $t = 10$ is the net premium reserve of a 10-pay semi-continuous whole life insurance of 120,000 level death benefit on (60) using a modified net level premium $\pi = 2,270.76$.

METHOD 2

At $t = 10$, the insurance contract at the attained age 60 consists of two policies: Policy One is a 10-pay semi-continuous whole life insurance of 100,000 level death benefit; Policy Two is a fully paid-up 20,000 continuous insurance.

$${}_{10}V_1^n = 100,000\overline{A}_{60} - \pi\ddot{a}_{60:\overline{10}|} = 100000 \times 1.06^{0.5} \times 0.36913 - 2270.76 \times 7.27894 = 21475.53$$

$${}_{10}V_2^n = 20,000\overline{A}_{60} = 20000 \times 1.06^{0.5} \times 0.36913 = 7600.85, \quad {}_{10}V^n = 21475.53 + 7600.85 = 29076$$

(d) The prospective reserve is different from the retrospective reserve. This is mainly because the premium basis is different from the reserving basis. The net level premium assumes no bonuses, but the prospective reserve at $t = 10$ includes the accrued bonuses 20,000 which are guaranteed for the future years.

$$(e) \quad {}_1V^{Retro} = \frac{({}_0V^{Retro} + \pi)(1+i) - 100000 \times 1.06^{0.5}q_{50}}{p_{50}} = \frac{(0 + 2270.76)1.06 - 100000 \times 1.06^{0.5} \times 0.00592}{0.99408} = 1808.21$$

$${}_2V^{Retro} = \frac{({}_1V^{Retro} + \pi)(1+i) - 102000 \times 1.06^{0.5}q_{51}}{p_{51}} = \frac{(1808.21 + 2270.76)1.06 - 102000 \times 1.06^{0.5} \times 0.00642}{0.99358} = 3673.09$$

$${}_3V^{Retro} = \frac{({}_1V^{Retro} + \pi)(1+i) - 104000 \times 1.06^{0.5}q_{52}}{p_{52}} = \frac{(3673.09 + 2270.76)1.06 - 104000 \times 1.06^{0.5} \times 0.00697}{0.99303} = 5593.16$$

$${}_1V^{Prosp} = 102000\overline{A}_{51} - \pi\ddot{a}_{51:\overline{19}|} = 102000 \times 1.06^{0.5} \times 0.25961 - 2270.76 \times 10.97432 = 2343.00$$

$${}_2V^{Prosp} = 104000\overline{A}_{52} - \pi\ddot{a}_{52:\overline{18}|} = 104000 \times 1.06^{0.5} \times 0.27050 - 2270.76 \times 10.64111 = 4800.26$$

$${}_3V^{Prosp} = 106000\overline{A}_{52} - \pi\ddot{a}_{53:\overline{17}|} = 106000 \times 1.06^{0.5} \times 0.28172 - 2270.76 \times 10.29134 = 7375.98$$

The retrospective reserve is smaller because it doesn't attempt to project future bonuses. If the premium basis and the reserve basis are the same and the premium is calculated using the equivalence principle, the retrospective reserve's blindness to future liabilities doesn't matter as this method produces the same reserve as does the prospective method. However, in this problem, the two bases are different and the retrospective backward-looking approach is no longer appropriate. In contrast, the prospective reserve method, when projecting future liabilities, at least considers the bonuses that have accrued as of the valuation date. The prospective reserve method better reflects future liabilities.

Example 27.2.4

Re-do the previous problem assuming that the annual bonuses are awarded at the beginning of the year, instead of the end of the year.

Solution 27.2.4

$$(a)\quad L_g^0 = 150 + 0.57P + \Big(100, 100 + 2,000(K_{[50]}+1)\Big)v^{K_{[50]}+1}(1+i)^{0.5} - 0.97P\ddot{a}_{\overline{\min(1+K_{[50]},20)}|}$$

$$(b)\quad E(L_g^0) = 0 : P = \frac{150 + 10100A_{[50]}(1+i)^{0.5} + 2000(1+i)^{0.5}(IA)_{[50]}}{0.97\ddot{a}_{[50]:\overline{20}|} - 0.57}$$

$$= \frac{150 + 100100 \times 0.18913(1.05^{0.5}) + 2000(1.05^{0.5})5.82662}{0.97 \times 12.84560 - 0.57} = \frac{31,490.06}{11.890227} = 2,648.40$$

(c) The net level premium is still $\pi = 2,270.76$. The prospective net premium reserve at $t = 10$ is still ${}_{10}V^n = 120000\overline{A}_{60} - \pi\ddot{a}_{60:\overline{10}|} = 29076$. Standing at $t = 10$, we see a 10-pay semi-continuous whole life insurance of 120,000 on (60), no matter the annual bonuses are awarded at the beginning or the end of the year.

(d) Same answer as before.

$$(e)\quad {}_1V^{Retro} = \frac{({}_0V^{Retro} + \pi)(1+i) - 102000 \times 1.06^{0.5}q_{50}}{p_{50}} = \frac{(0 + 2270.76)1.06 - 102000 \times 1.06^{0.5} \times 0.00592}{0.99408} = 1795.95$$

$${}_2V^{Retro} = \frac{({}_1V^{Retro} + \pi)(1+i) - 104000 \times 1.06^{0.5}q_{51}}{p_{51}} = \frac{(1795.95 + 2270.76)1.06 - 104000 \times 1.06^{0.5} \times 0.00642}{0.99358} = 3646.71$$

$${}_3V^{Retro} = \frac{({}_1V^{Retro} + \pi)(1+i) - 106000 \times 1.06^{0.5}q_{52}}{p_{52}} = \frac{(3646.71 + 2270.76)1.06 - 106000 \times 1.06^{0.5} \times 0.00697}{0.99303} = 5550.54$$

The prospective reserves remain unchanged.
The retrospective reserve is smaller when bonuses are added at the beginning of the year than bonuses are added at the end of the year, causing the gap between the prospective method and the retrospective method to become bigger. The prospective reserve is more prudent.

Example 27.2.5

For a special semi-continuous 20-pay with-profit whole insurance on (50), you are given:

- the initial insurance is 100,000
- death benefit is paid at the moment of death

The insurer prices the product on the following basis:

(i) Mortality: the Standard Select Survival Model

(ii) $i = 0.05$

(iii) Bonuses: compound bonus rate 2% per year. The bonus is added at the end of the year.

(iv) initial expenses: 150 plus 60% of Year 1 premium

(v) renewal expenses: 3% of the renewal premiums

(vi) claims expenses: 100 on death

(vii) EPV of continuous term insurance valuation method: claim acceleration approach

(viii) selective actuarial values: $A_{[50]} = 0.18913$, $\ddot{a}_{[50]:\overline{20}|} = 12.84560$, $A_{[50]j} = 0.35813$ where $j = 2.9411765\%$

What you need to do:

(a) Write down the expression for the gross premium loss-at-issue random variable

(b) Calculate the gross premium using the equivalence principle

(c) Ten years after issue, bonuses totaling $100,000(1.02^{10} - 1)$ have been declared. Calculate the prospective net premium reserve for the policy at $t = 10$ using the Illustrative Life Table and $i = 0.06$.

(d) Calculate the net premium reserve at $t = 1, 2, 3$ respectively using both the retrospective method and the prospective method.

Solution 27.2.5

$$(a)\quad L_g^0 = 150 + 0.57P + 100000(1.02^{K_{[50]}})v^{T_{[50]}} + 100v^{T_{[50]}} - 0.97P\ddot{a}_{\overline{\min(1+K_{[50]},20)}|}$$

$$\text{claim acceleration approach} \Rightarrow T_{[x]} = K_{[x]} + 0.5 = (K_{[x]} + 1) - 0.5$$

$$L_g^0 = 150 + 0.57P + 100000(1.02^{-1})(1.02^{K_{[50]}+1})v^{K_{[50]}+1}(1+i)^{0.5} + 100v^{K_{[50]}+1}(1+i)^{0.5} - 0.97P\ddot{a}_{\overline{\min(1+K_{[50]},20)}|}$$

$$= 150 + 0.57P + 100000(1.02^{-1})v_j^{K_{[50]}+1}(1+i)^{0.5} + 100v^{K_{[50]}+1}(1+i)^{0.5} - 0.97P\ddot{a}_{\overline{\min(1+K_{[50]},20)}|}$$

$$v_j = 1.02v = \frac{1.02}{1+i}, \quad j = \frac{1+i}{1.02} - 1 = \frac{0.05-0.02}{1.02} = 2.9411765\%$$

$$(b) \quad E(L_g^0) = 0: \quad 150 + 0.57P + 100000(1.02^{-1})A_{[50]j}(1+i)^{0.5} + 100A_{[50]}(1+i)^{0.5} - 0.97P\ddot{a}_{[50]:\overline{20}|} = 0$$

$$P = \frac{150 + 100000(1.02^{-1})A_{[50]j}(1+i)^{0.5} + 100A_{[50]}(1+i)^{0.5}}{0.97\ddot{a}_{[50]:\overline{20}|} - 0.57}$$

$$= \frac{150 + 100000(1.02^{-1}) \times 0.35813 \times 1.05^{0.5} + 100 \times 0.18913(1.05^{0.5})}{0.97 \times 12.84560 - 0.57} = \frac{36,147.50}{11.890227} = 3,040.10$$

(c) **METHOD 1**

$${}_{10}V^n = 100000(1.02^{10})\bar{A}_{60} - \pi\ddot{a}_{60:\overline{10}|} = 100000(1.02^{10}) \times 1.06^{0.5}A_{60} - \pi\ddot{a}_{60:\overline{10}|}$$

$$= 100000(1.02^{10}) \times 1.06^{0.5} \times 0.36913 - 2270.76 \times 7.27894 = 29798.26$$

The net premium reserve at $t = 10$ is the net premium reserve of a 10-pay semi-continuous whole life insurance of $100000(1.02^{10})$ level death benefit on (60) using a modified net level premium $\pi = 2,270.76$.

METHOD 2

At $t = 10$, the insurance contract at the attained age 60 consists of two policies: Policy One is a 10-pay semi-continuous whole life insurance of 100,000 level death benefit; Policy Two is a fully paid-up $100000(1.02^{10} - 1)$ continuous insurance.

$${}_{10}V_1^n = 100,000\bar{A}_{60} - \pi\ddot{a}_{60:\overline{10}|} = 100000 \times 1.06^{0.5} \times 0.36913 - 2270.76 \times 7.27894 = 21475.53$$

$${}_{10}V_2^n = 100000(1.02^{10} - 1)\bar{A}_{60} = 100000(1.02^{10} - 1) \times 1.06^{0.5} \times 0.36913 = 8322.72$$

$${}_{10}V^n = 21475.53 + 8322.72 = 29798.25$$

$$(d) \quad {}_1V^{Retro} = \frac{({}_0V^{Retro} + \pi)(1+i) - 100000 \times 1.06^{0.5}q_{50}}{p_{50}} = \frac{(0 + 2270.76)1.06 - 100000 \times 1.06^{0.5} \times 0.00592}{0.99408} = 1808.21$$

$${}_2V^{Retro} = \frac{({}_1V^{Retro} + \pi)(1+i) - 100000 \times 1.02 \times 1.06^{0.5}q_{51}}{p_{51}}$$

$$= \frac{(1808.21 + 2270.76)1.06 - 100000 \times 1.02 \times 1.06^{0.5} \times 0.00642}{0.99358} = 3673.09$$

$${}_3V^{Retro} = \frac{({}_1V^{Retro} + \pi)(1+i) - 100000 \times 1.02^2 \times 1.06^{0.5}q_{52}}{p_{52}}$$

$$= \frac{(3673.09 + 2270.76)1.06 - 100000 \times 1.02^2 \times 1.06^{0.5} \times 0.00697}{0.99303} = 5592.87$$

$${}_1V^{Prosp} = 100000 \times 1.02\bar{A}_{51} - \pi\ddot{a}_{51:\overline{19}|} = 100000 \times 1.02 \times 1.06^{0.5} \times 0.25961 - 2270.76 \times 10.97432 = 2343.00$$

$${}_2V^{Prosp} = 100000 \times 1.02^2\bar{A}_{52} - \pi\ddot{a}_{52:\overline{18}|} = 100000 \times 1.02^2 \times 1.06^{0.5} \times 0.27050 - 2270.76 \times 10.64111 = 4811.40$$

$${}_3V^{Prosp} = 100000 \times 1.02^3\bar{A}_{52} - \pi\ddot{a}_{53:\overline{17}|} = 100000 \times 1.02^3 \times 1.06^{0.5} \times 0.28172 - 2270.76 \times 10.29134 = 7411.01$$

Example 27.2.6

Same as the last problem EXCEPT the annual bonuses are awarded at the beginning of the year.
What you need to do:

(a) Write down the expression for the gross premium loss-at-issue random variable

(b) Calculate the gross premium using the equivalence principle.

Solution 27.2.6

$$(a) \quad L_g^0 = 150 + 0.57P + 100000(1.02^{K_{[50]}+1})v^{T_{[50]}} + 100v^{T_{[50]}} - 0.97P\ddot{a}_{\overline{\min(1+K_{[50]},20)}|}$$

$$= 150 + 0.57P + 100000(1.02^{K_{[50]}+1})v^{K_{[50]}+1}(1+i)^{0.5} + 100v^{K_{[50]}+1}(1+i)^{0.5} - 0.97P\ddot{a}_{\overline{\min(1+K_{[50]},20)}|}$$

$$= 150 + 0.57P + 100000v_j^{K_{[50]}+1}(1+i)^{0.5} + 100v^{K_{[50]}+1}(1+i)^{0.5} - 0.97P\ddot{a}_{\overline{\min(1+K_{[50]},20)}|}$$

$$v_j = 1.02v = \frac{1.02}{1+i}, \quad j = \frac{1+i}{1.02} - 1 = \frac{0.05-0.02}{1.02} = 2.9411765\%$$

$$(b) \quad P = \frac{150 + 100000A_{[50]j}(1+i)^{0.5} + 100A_{[50]}(1+i)^{0.5}}{0.97\ddot{a}_{[50]:\overline{20}|} - 0.57}$$

$$= \frac{150 + 100000 \times 0.35813 \times 1.05^{0.5} + 100 \times 0.18913(1.05^{0.5})}{0.97 \times 12.84560 - 0.57} = \frac{36866.785}{11.890227} = 3100.59$$

27.3 check your knowledge

Homework 27.3.1

For fully discrete participating whole life insurance on (50) with an initial insurance of 100,000, you are given the following premium basis:

(i) Mortality: the Illustrative Life Table

(ii) Simple reversionary bonus: 3% per year added at the beginning of year (including Year 1)

(iii) $i = 0.06$

(iv) The following expenses are payable at the beginning of the year:

	Percent of Premium	Per Policy
First Year	25%	200
Renewal	10%	50

(v) Selective actuarial value: $(IA)_{51} = 5.06254$

(vi) Premiums are determined using the equivalence principle.

- Write the expression for the gross premium loss at issue random variable L_g^0 and the net premium loss at issue random variable L_n^0
- Calculate the level annual gross premium P^g and the level annual net premium P^n

Homework Solution 27.3.1

★★★★☆ Difficulty

	Percent of Premium	Per Policy
All years	10%	50
First Year extra	15%	150

$$L_g^0 = 100000\left(1 + 0.03(K_{50}+1)\right)v^{K_{50}+1} + 0.1P^g\ddot{a}_{\overline{K_{50}+1}|} + 0.15P^g + 50\ddot{a}_{\overline{K_{50}+1}|} + 150 - P^g\ddot{a}_{\overline{K_{50}+1}|}$$

$$P^g = \frac{100000A_{50} + 100000 \times 0.03(IA)_{50} + 50\ddot{a}_{50} + 150}{0.9\ddot{a}_{50} - 0.15}$$

$$= \frac{100000 \times 0.24905 + 100000 \times 0.03(4.99676) + 50 \times 13.26683 + 150}{0.9 \times 13.26683 - 0.15} = 3452.74$$

$$(IA)_{50} = A_{50} + {}_1E_{50}(IA)_{51} = 0.24905 \times 0.937811 + 5.06254 = 4.99676$$

$$L_n^0 = 100000v^{K_{50}+1} - P^n\ddot{a}_{\overline{K_{50}+1}|}, \quad P^n = \frac{100000A_{50}}{\ddot{a}_{50}} = \frac{100000 \times 0.24905}{13.26683} = 1877.24$$

Homework 27.3.2

For a special fully discrete with-profit whole insurance on (x), you are given:

- initial insurance: $100,000$

The insurer prices the product on the following basis:

(i) Mortality: the Illustrative Life Table

(ii) $i = 0.06$

(iii) Bonuses: compound bonus rate 2.5% per year. The bonus is added at the beginning of the year.

(iv) initial expenses — per policy: 100 plus 30% of Year 1 premium

(v) renewal expenses — per policy: 20 plus 4% of the renewal premiums

(vi) claims expenses: 50 per claim

What you need to do:

(a) Write down the expression for the gross premium loss-at-issue random variable

(b) Write down the expression for the gross premium using the equivalence principle

Homework Solution 27.3.2

★★★★☆ Difficulty

PV

- total insurance $100,000(1.025)^{K_x+1}v^{K_x+1}=100,000v_j^{K_x+1}, j=\dfrac{1.06}{1.035}-1$
- claim cost $50v^{K_x+1}$
- premium expense $(0.04\ddot{a}_{\overline{K_x+1}|}+0.26)P$
- per policy expense $20\ddot{a}_{\overline{K_x+1}|}+80$
- total premiums $P\ddot{a}_{\overline{K_x+1}|}$

EPV

- death benefit $100,000A_{xj}$
- claim cost $50A_x$
- premium expense $(0.04\ddot{a}_x+0.26)P$
- per policy expense $20\ddot{a}_x+80$
- total premiums $P\ddot{a}_x$

$$L_g^0=100,000v_j^{K_x+1}+50v^{K_x+1}+20\ddot{a}_{\overline{K_x+1}|}+80-P(0.96\ddot{a}_{\overline{K_x+1}|}-0.26),\quad P=\frac{100,000A_{xj}+50A_x+20\ddot{a}_x+80}{0.96\ddot{a}_x-0.26}$$

Homework 27.3.3

For a 3-year fully discrete with-profit term insurance on (42), you are given:

- initial insurance: 10000

The insurer prices the product on the following basis:

(i) Mortality: the Illustrative Life Table

(ii) $i=0.06$

(iii) Bonuses: super compound reversionary bonus with $\alpha=2\%$ and $\beta=5\%$ per year. The bonus is vested at the end of the year.

(iv) expenses: 5% of each premium

Calculate the gross premium using the equivalence principle.

Homework Solution 27.3.3

★★★☆☆ Difficulty

Death benefit for each year:

$$DB_1=10000,\quad DB_2=10000(1+1\times0.02)=10200,\quad DB_3=10000(1+2\times0.02)+200\times0.05=10410$$

$$P=\frac{10000d_{42}v+10200d_{43}v^2+10410d_{43}v^3}{0.95(\ell_{42}+\ell_{43}v+\ell_{43}v^2)}=\frac{10000(29646)(1.06^{-1})+10200(31776)(1.06^{-2})+10410(34098)(1.06^{-3})}{0.95(9259571+9229925(1.06^{-1})+9198149(1.06^{-2}))}=34.86$$

Homework 27.3.4

For a special fully discrete 15-year with-profit endowment insurance policy on (35) with the initial sum insured of 100,000, you are given the following gross premium basis:

(i) Mortality: the Illustrative Life Table

(ii) $i=0.06$

(iii) Premium: level gross monthly premium P calculated using the equivalence principle

(iv) The insurer declares a simple reversionary bonus 4% per year. The bonus is added at the end of the year

(v) initial expenses: $120+0.5(12P)$, that is, a fixed cost 120 plus 50% of Year 1 total gross premiums

(vi) renewal expenses: 5% of the second and subsequent monthly premiums

(vii) claims expenses: 200 on death; 100 on maturity

Selective actuarial values:

- $A_{\overset{1}{35}:\overline{15}|}=0.029982$, ${}_{15}E_{35}=0.396458$, $\ddot{a}_{35:\overline{15}|}=10.13288$, $(IA)_{\overset{1}{35}:\overline{15}|}=0.24550$
- $A_{\overset{1}{35}:\overline{5}|}=0.009544$, ${}_5E_{35}=0.738732$, $\ddot{a}_{35:\overline{5}|}=4.44713$, $(IA)_{\overset{1}{35}:\overline{5}|}=0.02869$, ${}_5p_{35}=0.988590$
- $A_{\overset{1}{40}:\overline{10}|}=0.027667$, ${}_{10}E_{40}=0.536674$, $\ddot{a}_{40:\overline{10}|}=7.69664$

What you need to do:

(a) Use the Woolhouse formula, calculate $\ddot{a}^{(12)}_{35:\overline{15}|}$.

(b) Calculate the monthly gross premium P.

(c) 5 years after issue and immediately before the then premium due and immediately after the bonus is awarded, the insured plans to surrender the policy. The surrender value is equal 80% of the gross premium retrospective policy value. The gross premium policy value is calculated on the same basis as the gross premium basis. The insurer has declared the annual 4% reversionary bonus at the end of each year for the first 5 years consistent with the gross premium assumption. Calculate the surrender value at $t=5$.

(d) Suppose that the surrender value is equal to 80% of the prospective gross premium policy, as opposed to the 80% of the retrospective gross premium policy value. The prospective policy value basis is the same as the gross premium basis EXCEPT the insurer plans to use a reduced simple reversionary rate of 2% for the remaining 10 years. Calculate the revised surrender value at $t = 5$.

(e) Verify that if the bonus rate were 4% per year consistent with the gross premium basis, then the surrender value in (c) and (d) would be equal.

(f) Suppose that the surrender value is equal to 80% of the net premium reserve. Calculate the revised surrender value at $t = 5$ under the following net premium reserve basis:

- Mortality: Select Survival Model
- $i = 0.05$
- At the end of Year 5, bonuses totaling 20,000 have been declared.
- Premium: annual level premium
- Selective actuarial values: ${}_5V_{35:\overline{15}|} = 0.255699$, $A_{\overset{1}{40}:\overline{10}|} = 0.005732$

Homework Solution 27.3.4

★★★★★ Difficulty

EPV of each item:

- monthly gross premiums: $12P\ddot{a}^{(12)}_{35:\overline{15}|}$
- initial death benefits: $100,000A_{\overset{1}{35}:\overline{15}|}$
- annual bonuses: $4,000(IA)_{\overset{1}{35}:\overline{15}|} - 4,000A_{\overset{1}{35}:\overline{15}|}$
- maturity benefit: $100,000(1 + 0.04 \times 15)_{15}E_{35} = 160000_{15}E_{35}$
- death claim expense: $200A_{\overset{1}{35}:\overline{15}|}$
- maturity claim expense: $100_{15}E_{35}$
- initial expense: $120 + 0.5(12P)$
- renewal expense: $(0.05)12P\ddot{a}^{(12)}_{35:\overline{15}|} - 0.05P$

$$(a),(b) \quad \left((0.95)12\ddot{a}^{(12)}_{35:\overline{15}|} - 0.5(12) + 0.05\right) P = (100000-4000+200)A_{\overset{1}{35}:\overline{15}|} + 4000(IA)_{\overset{1}{35}:\overline{15}|} + (160000+100)_{15}E_{35} + 120$$

$$\ddot{a}^{(12)}_{35:\overline{15}|} = \ddot{a}_{35:\overline{15}|} - \frac{11}{24}(1 - {}_{15}E_{35}) = 10.13288 - \frac{11}{24}(1 - 0.396458) = 9.856255$$

$$P = \frac{(100000 - 4000 + 200)0.029982 + 4000 \times 0.24550 + (160000 + 100)0.396458 + 120}{(0.95)12 \times 9.856255 - 0.5(12) + 0.05} = \frac{67459.194}{106.41131} = 633.95$$

$$(c) \quad {}_5V^{Retro} = \frac{1.06^5}{{}_5p_{35}}\left((0.95 \times 12\ddot{a}^{(12)}_{35:\overline{5}|} - 0.5 \times 12 + 0.05)P - (100000 - 4000 + 200)A_{\overset{1}{35}:\overline{5}|} - 4000(IA)_{\overset{1}{35}:\overline{5}|} - 120\right)$$

$$= \frac{1.06^5}{0.988590}\left((0.95 \times 12 \times 4.32738 - 0.5 \times 12 + 0.05)633.95 - (96200)0.009544 - 4000 \times 0.02869 - 120\right) = 35,668.02$$

$$\ddot{a}^{(12)}_{35:\overline{5}|} = \ddot{a}_{35:\overline{5}|} - \frac{11}{24}(1 - {}_5E_{35}) = 4.44713 - \frac{11}{24}(1 - 0.738732) = 4.32738$$

$$0.8(35668.02) = 28534.42$$

(d) 5 years after issue, if the policy is still in force, we see a 10-year endowment policy on (40). The policy has an accrued bonus of 20,000 and will declare an annual bonus of 2,000 each year for the next 10 years. EPVs are:

- monthly gross premiums: $12P\ddot{a}^{(12)}_{40:\overline{10}|}$
- initial benefits: $100,000A_{\overset{1}{40}:\overline{10}|}$
- accrued bonuses: $20,000A_{\overset{1}{40}:\overline{10}|}$
- future bonuses: $2,000(IA)_{\overset{1}{40}:\overline{10}|} - 2,000A_{\overset{1}{40}:\overline{10}|}$
- maturity benefit: $(120,000 + 20,000)_{10}E_{40}$
- death claim expense: $200A_{\overset{1}{40}:\overline{10}|}$
- maturity claim expense: $100_{10}E_{40}$
- renewal expense: $(0.05)12P\ddot{a}^{(12)}_{40:\overline{10}|}$

$${}_5V = (100,000+20,000+200-2,000)A_{\overset{1}{40}:\overline{10}|} + 2,000(IA)_{\overset{1}{40}:\overline{10}|} + (100,000+20,000+20,000+100)_{10}E_{40} - (0.95)12P\ddot{a}^{(12)}_{40:\overline{10}|}$$

$$= (118,200)A_{\overset{1}{40}:\overline{10}|} + 2,000(IA)_{\overset{1}{40}:\overline{10}|} + 140,100_{10}E_{40} - (0.95)12P\ddot{a}^{(12)}_{40:\overline{10}|}$$

$$\ddot{a}^{(12)}_{40:\overline{10}|} = \ddot{a}_{40:\overline{10}|} - \frac{11}{24}(1 - {}_{10}E_{40}) = 7.69664 - \frac{11}{24}(1 - 0.536674) = 7.4842823$$

$$(IA)_{\overset{1}{35:\overline{15}|}} = (IA)_{\overset{1}{35:\overline{5}|}} + {}_5E_{35}\Big((IA)_{\overset{1}{40:\overline{10}|}} + 5A_{\overset{1}{40:\overline{10}|}}\Big)$$

$$0.24550 = 0.02869 + 0.738732\Big((IA)_{\overset{1}{40:\overline{10}|}} + 5\times 0.027667\Big), \quad (IA)_{\overset{1}{40:\overline{10}|}} = 0.15515$$

$${}_5V = 118200\times 0.027667 + 2000\times 0.15515 + 140100\times 0.536674 - (0.95)12\times 633.95\times 7.22216 = 26573.80$$

$$0.8(26573.80) = 21259$$

The prospective policy value is lower because the annual bonus rate for the remaining 10 years is lower than what is assumed in the premium basis.

(e) 5 years after issue, if the policy is still in force, we see a 10-year endowment policy on (40). The policy has an accrued bonus of 20,000 and will declare an annual bonus of 4,000 each year for the next 10 years. EPVs are:

- monthly gross premiums: $12P\ddot{a}^{(12)}_{40:\overline{10}|}$
- initial benefits: $100,000A_{\overset{1}{40:\overline{10}|}}$
- accrued bonuses: $20,000A_{\overset{1}{40:\overline{10}|}}$
- future bonuses: $4,000(IA)_{\overset{1}{40:\overline{10}|}} - 4,000A_{\overset{1}{40:\overline{10}|}}$
- maturity benefit: $(120,000+40,000)_{10}E_{40}$
- death claim expense: $200A_{\overset{1}{40:\overline{10}|}}$
- maturity claim expense: $100_{10}E_{40}$
- renewal expense: $(0.05)12P\ddot{a}^{(12)}_{40:\overline{10}|}$

$${}_5V = (120,000+200-4,000)A_{\overset{1}{40:\overline{10}|}} + 4,000(IA)_{\overset{1}{40:\overline{10}|}} + 160,100_{10}E_{40} - (0.95)12P\ddot{a}^{(12)}_{40:\overline{10}|}$$

$$= 116200\times 0.027667 + 4000\times 0.15515 + 160100\times 0.536674 - (0.95)12\times 633.95\times 7.484283 = 35668.0 = {}_5V^{Retro}$$

(f) The net premium ignores any bonuses because bonuses are not guaranteed at issue. However, at $t=5$, the accrued bonuses 20,000 are guaranteed for the remainder of the contract. The reserve at $t=5$ is the net premium reserve of a 10-year endowment insurance of 120,000 on (40) using the net level premium that is equal to the net level premium of a 15-year endowment insurance of 100,000 on (35).

$$NP = 100000\frac{A_{35:\overline{15}|}}{\ddot{a}_{35:\overline{15}|}} = 100000P_{35:\overline{15}|}$$

$${}_5V = 120000A_{40:\overline{10}|} - 100000P_{35:\overline{15}|}\ddot{a}_{40:\overline{10}|} = \Big(100000A_{40:\overline{10}|} - 100000P_{35:\overline{15}|}\ddot{a}_{40:\overline{10}|}\Big) + 20000A_{40:\overline{10}|}$$

$$= 100000{}_5V_{35:\overline{15}|} + 20000A_{\overset{1}{40:\overline{10}|}} = 100000\times 0.255699 + 20000(0.005732) = 25684.54$$

$$0.8\times 25684.54 = 20547.63$$

Notation:

- $P_{35:\overline{15}|}$ is the net level premium of a 15-year endowment insurance of 1 on (35)
- ${}_5V_{35:\overline{15}|}$ is the net level premium reserve at $t=5$ of a 15-year endowment insurance of 1 on (35)

Homework 27.3.5

An insurer issues a fully discrete 5-year with profit endowment insurance policy on (40). The policy has an initial insurance amount of 20,000. Simple reversionary bonuses are added at the beginning of each year including Year 1.
(a) Show that the annual gross premium is 4,769 using the equivalence principle. Basis:

- Mortality: Standard Select Survival Table
- $i = 0.05$
- Initial expenses: 50% of the first premium
- Renewal expenses: 5% of the renewal premiums
- Bonus: Simple 4% per year added at the beginning of the year
- Selective actuarial values: $A_{\overset{1}{[40]:\overline{4}|}} = 0.002535$, $(IA)_{\overset{1}{[40]:\overline{4}|}} = 0.00791$, ${}_5E_{[40]} = 0.781208$

(b) Calculate the net premium reserve at $t = 0,1,2,3,4$. Basis:

- Mortality: Illustrative Life Table
- $i = 0.06$

Homework Solution 27.3.5

★★★★☆ Difficulty

$$(a)\quad P\ddot{a}_{[40]:\overline{5}|} = 20,000A_{[40]:\overline{5}|} + 800(IA)_{[40]:\overline{5}|} + 0.05P\ddot{a}_{[40]:\overline{5}|} + 0.45P$$

$$A_{[40]:\overline{5}|} = 0.002535+0.781208 = 0.783743, (IA)_{[40]:\overline{5}|} = 0.00791+5\times0.781208 = 3.91395, \ddot{a}_{[40]:\overline{5}|} = \frac{1-0.783743}{0.05/1.05} = 4.54140$$

$$P = \frac{20,000A_{[40]:\overline{5}|} + 800(IA)_{[40]:\overline{5}|}}{0.95\ddot{a}_{[40]:\overline{5}|} - 0.45} = \frac{20000\times0.783743 + 800\times3.91395}{0.95\times4.54140 - 0.45} = 4866.57$$

(b) The net premium ignores future bonuses as they are not guaranteed at issue. However, the accrued bonuses as of the valuation date are guaranteed and should be counted as future benefits.

METHOD 1

$$NP = 200,00\frac{A_{40:\overline{5}|}}{\ddot{a}_{40:\overline{5}|}} = 20,000P_{40:\overline{5}|} = 20000\times\frac{0.748675}{4.44007} = 3372.35$$

$${}_0V = 0$$

One year after issue, the accrued bonus 800 is guaranteed and we see a 4-year endowment insurance of 20,800 on (41).

$${}_1V = 20800A_{41:\overline{4}|} - NP\ddot{a}_{41:\overline{4}|} = 20800\times0.793020 - 3372.35\times3.65665 = 4,163.30$$

$${}_2V = 21600A_{42:\overline{3}|} - NP\ddot{a}_{42:\overline{3}|} = 21600\times0.840124 - 3372.35\times2.82447 = 8,621.58$$

$${}_3V = 22400A_{43:\overline{2}|} - NP\ddot{a}_{43:\overline{2}|} = 22400\times0.890180 - 3372.35\times1.94015 = 13,397.17$$

$${}_4V = 23200A_{44:\overline{1}|} - NP\ddot{a}_{44:\overline{1}|} = 23200\times0.943396 - 3372.35\times1 = 18,514.44$$

${}_{5^-}V = 24000$. Time 5^- means immediately before the maturity payment of 24,000 is made.

METHOD 2. The logic is similar to Part (f) of the last problem.

$${}_0V = 0$$

$${}_1V = 20000\,{}_1V_{40:\overline{5}|} + 800A_{41:\overline{4}|} = 20000\left(1-\frac{\ddot{a}_{41:\overline{4}|}}{\ddot{a}_{40:\overline{5}|}}\right) + 800A_{41:\overline{4}|} = 20000\left(1-\frac{3.65665}{4.44007}\right) + 800\times0.793020 = 4,163.30$$

$${}_2V = 20000\,{}_2V_{40:\overline{5}|} + 1600A_{42:\overline{3}|} = 20000\left(1-\frac{\ddot{a}_{42:\overline{3}|}}{\ddot{a}_{40:\overline{5}|}}\right) + 800A_{41:\overline{4}|} = 20000\left(1-\frac{2.82447}{4.44007}\right) + 1600\times0.840124 = 8,621.58$$

$${}_3V = 20000\,{}_3V_{40:\overline{5}|} + 2400A_{43:\overline{2}|} = 20000\left(1-\frac{\ddot{a}_{43:\overline{2}|}}{\ddot{a}_{40:\overline{5}|}}\right) + 2400A_{43:\overline{2}|} = 20000\left(1-\frac{1.94015}{4.44007}\right) + 2400\times0.890180 = 13,397.17$$

$${}_4V = 20000\,{}_4V_{40:\overline{5}|} + 3200A_{44:\overline{1}|} = 20000\left(1-\frac{\ddot{a}_{44:\overline{1}|}}{\ddot{a}_{40:\overline{5}|}}\right) + 3200A_{44:\overline{1}|} = 20000\left(1-\frac{1}{4.44007}\right) + 3200\times0.943396 = 18,514.44$$

$${}_{5^-}V = 20000\,{}_{5^-}V_{40:\overline{5}|} + 4000 = 20000\left(1-\frac{0}{\ddot{a}_{40:\overline{5}|}}\right) + 4000 = 24000$$

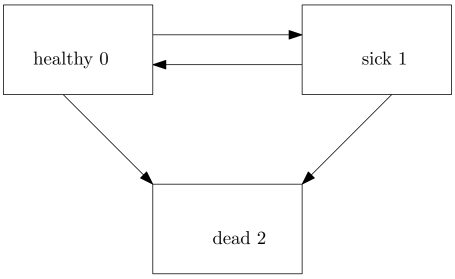
Chapter 29

Multiple state model

29.1 introduction

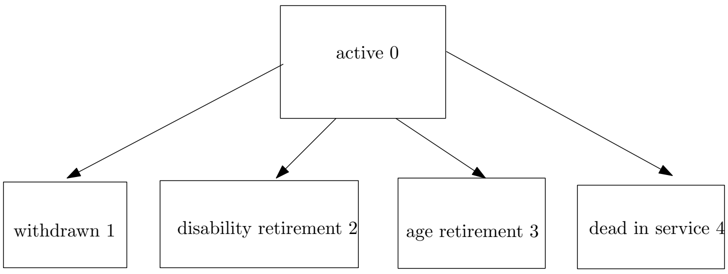
So far our life insurance and annuity contracts have only two major states: the insured (x) is either alive or dead. In this simple alive-dead model, the death benefit is triggered when (x) moves from the alive state to the dead state and the annuity payment is triggered when (x) remains in the alive state. Now we'll consider contracts where cash flows are triggered when (x) moves from one of the multiple states to the same or a different state after a time interval.

Example 29.1.1



A 5-year combined death and sickness policy is issued to healthy lives aged 45. Annual premiums are payable continuously by healthy policyholders. The policy pays 100,000 immediately on death, with an additional 20,000 if the deceased is sick at the time of death. There is also a benefit of 5,000 per year payable continuously to sick policyholders.

Example 29.1.2



An active member in a pension plan can

- withdraw from the plan due to job change and receives no benefit
- becomes disabled before the normal retirement and receives a lump sum of 80,000
- retires at the normal retirement age 65 and receives an annual pension benefit equal to 3% of the average career salary for each year of service, or
- dies while in service and receives a lump sum of 100,000

29.2 notation

transition probability

We consider actuarial problems where cash flows depend on the continuation or the change of an individual's state. An individual can be in a finite set of $n+1$ states labelled $0, 1, \ldots, n$, with instantaneous transitions being possible between selected pairs of states. The state random variable $Y(t)$ takes one of the values $0, 1, \ldots, n$. The event $Y(t) = i$ for $t \geq 0$ means that the individual is in state i at age $x+t$. $Y(t)$ is called a stochastic process. Any random variable indexed by time is a stochastic process. For example, let $X(t)$ represents the temperature of a location at time t. If temperatures are recorded continuously or at regular intervals, the stream of temperatures is a stochastic process.

${}_tp_x^{ij} = P\big[Y(x+t) = j \mid Y(x) = i\big]$ is the probability that $(x+t)$ is in state j given that (x) is in state i. This definition implies that the state at $x+t$ only depends on the state at x and not on the state history up to t. This is called the Markov property, which assumes that the conditional probability distribution of future states depends only on the present state, not on the sequence of states that precede the present state. If the process $Y(t)$ meets the Markov property, this process is called the Markov process. Otherwise, the process is called the non-Markov process.

In some actuarial applications, the stream of states doesn't meet the Markov property. For example, whether a person will be disabled 6 months from today may depend on whether he's disabled today as well as on his disability history in the past. As another example, the select mortality is not purely a function of the attained age but it depends on when an individual is selected. However, the AMLCR textbook focuses on a Markov process.

Next, you need to understand the difference between ${}_tp_x^{ii}$ and ${}_tp_x^{\overline{ii}}$. In both symbols, (x) is in state i now and in state i t years from now. However, $\overline{ii}$ means being continuously in state i during the time interval $[0, t]$, whereas ii allows for the individual to be in other states during $(0, t)$ as long as he returns to the state i at time t. $\overline{ii}$ is like being grounded at the state i, while ii is free to go anywhere (including staying put) as long as you return to your original state at the end of the day. Clearly, $\overline{ii}$ is a special case of ii and ${}_tp_x^{ii} \geq {}_tp_x^{\overline{ii}}$.

Example 29.2.1

In the basic survival model, ${}_2p_x = 0.9$. In the alive-dead model, calculate ${}_2p_x^{00}$, ${}_2p_x^{01}$, ${}_2p_x^{10}$, ${}_2p_x^{11}$, ${}_2p_x^{\overline{00}}$, ${}_2p_x^{\overline{11}}$.

Solution 29.2.1

${}_2p_x^{00} = {}_2p_x^{\overline{00}} = {}_2p_x = 0.9$ (once you leave state 1 and can never go back in the alive-death model).

${}_2p_x^{11} = {}_2p_x^{\overline{11}} = 1$ (once dead, always dead).

${}_2p_x^{01} = {}_2q_x = 0.1$, ${}_2p_x^{10} = 0$ (can't change from being dead to alive),

force of transition or transition intensity

If the state variable $Y(t)$ is continuous, then $\mu_x^{ij} = \lim_{h \to 0} \frac{{}_hp_x^{ij}}{h}$ for $i \neq j$ is called the force of transition or the transition intensity between state i and state j for age x. This is the counterpart of the force of mortality in the basic alive-dead model and $\mu_x^{01} = \mu_x$.

Example 29.2.2

In the alive(0)-dead(1) model, $\mu_x^{01} = \lim_{h \to 0} \frac{{}_hp_x^{01}}{h} = \lim_{h \to 0} \frac{{}_hq_x}{h}$.
Explain why μ_x^{01} is the force of mortality μ_x.

Solution 29.2.2

See section 5.1.

Another way to express $\mu_x^{ij} = \lim_{h \to 0} \frac{{}_hp_x^{ij}}{h}$ is ${}_hp_x^{ij} = h\mu_x^{ij} + o(h)$, where $o(h)$ is a function that approaches zero faster than h approaches zero. And for a small h, ${}_hp_x^{ij} \approx h\mu_x^{ij}$.

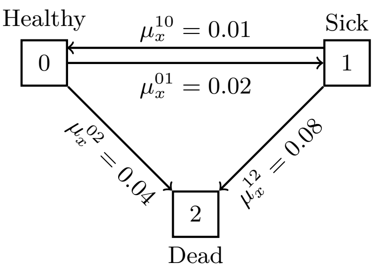
29.3 find probability of being stuck in a state

${}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j=0;j\neq i}^{n} \mu_{x+s}^{ij} ds \right)$. The term $\sum_{j=0;j\neq i}^{n} \mu_{x+s}^{ij}$ is the total transition forces leaving state i; the transitions entering state i are irrelevant. Learners tend to ask two questions: (1) "Why don't the transitions entering state i matter?" Answer: If we allow other states to flow into state i at any point during the interval $[0, t]$, the insured won't be continuously in state i any more. And (2) "If we want the insured to be continuously in state i, then why the term $\sum_{j=0;j\neq i}^{n} \mu_{x+s}^{ij}$?" Answer: the forces that push the insured off the state i cause the state i population to dwindle exponentially.

To help eliminate errors and correctly write the formula for ${}_tp_x^{\overline{ii}}$, change the multiple state diagram as follows: (1) keep only the i state and delete all the other states, (2) keep only the arrows leaving state i and delete all the arrows flowing into state i. Then sum up all the transition intensities in the new diagram and add a minus sign. The negative sign is needed because all the arrows leaving state i are working again state i. By the way, if you forget the minus sign in $-\int_0^t \sum_{j=0;j\neq i}^{n} \mu_{x+s}^{ij} ds$, pretty soon you'll find that ${}_tp_x^{\overline{ii}}$ will be greater than one.

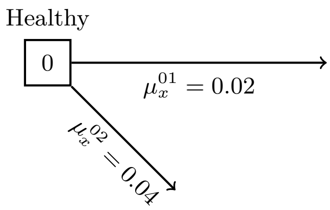
Example 29.3.1

A 10-year sickness policy is issued to a healthy life age 50. The policy pays a no-claim bonus of 1000 at the end of Year 10 if the insured remains healthy throughout the term of the contract. The transition intensities are constants for all ages. $\delta = 0.06$. Calculate the EPV of the bonus.



Solution 29.3.1

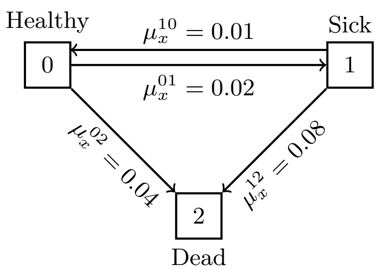
The probability of receiving the bonus is ${}_{10}p_{50}^{\overline{00}}$.



${}_{10}p_{50}^{\overline{00}} = \exp\left(-\int_0^{10} (\mu_{50+s}^{01} + \mu_{50+s}^{02}) ds \right) = \exp\left(-\int_0^{10} (0.02 + 0.04) ds \right) = e^{-0.06(10)} = e^{-0.6}$

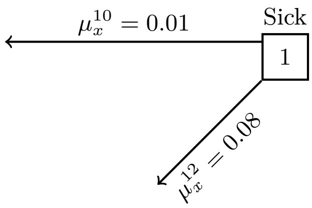
EPV: $1000 {}_{10}p_{50}^{\overline{00}} e^{-10\delta} = 1000e^{-0.12} = 886.92$

Example 29.3.2



The transition intensities are constants for all ages. Calculate the probability that (x) remains sick throughout the next 3 years given that he's sick today.

Solution 29.3.2



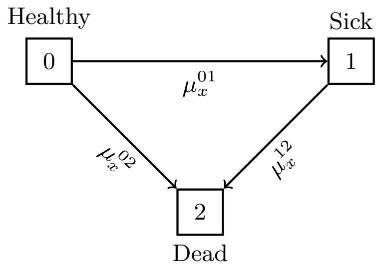
$_3p_x^{\overline{11}} = \exp\left(-\int_0^3 (\mu_{x+s}^{10} + \mu_{x+s}^{12})ds\right) = e^{-3(0.09)} = 0.7634$

29.4 when getting back to a state is the same as being stuck in the state

There are 2 sufficient conditions for $_tp_x^{ii} = {}_tp_x^{\overline{ii}}$: (1) a state receives arrows but doesn't send arrows, and (2) a state sends arrows but doesn't receive arrows. If a state both receives arrows and sends arrows, however, it's not clear whether $_tp_x^{ii} \neq {}_tp_x^{\overline{ii}}$ and you'll need to test whether the insured can reenter state i after he leaves state i.

Example 29.4.1

For which states does the equation $_tp_x^{ii} = {}_tp_x^{\overline{ii}}$ hold?



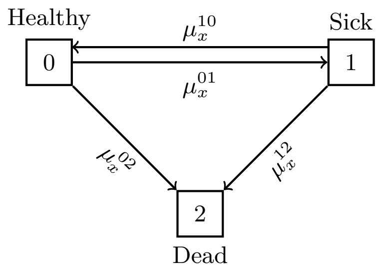
Solution 29.4.1

The "healthy" state sends out arrows but doesn't receive arrows (there's no re-entry into the "healthy" state). The "dead" state receives arrows and but doesn't send arrows (once in and never out) – this state is called the absorbing state. For these two states, the equation $_tp_x^{ii} = {}_tp_x^{\overline{ii}}$ holds and we have $_tp_x^{00} = {}_tp_x^{\overline{00}}$ and $_tp_x^{22} = {}_tp_x^{\overline{22}} = 1$. For any absorbing state i, $_tp_x^{ii} = {}_tp_x^{\overline{ii}} = 1$.

The "sick" state receives an arrow and emits an arrow. Does $_tp_x^{22} = {}_tp_x^{\overline{22}}$ hold? Surprisingly Yes. Once you leave the "sick" state, you can never reenter it.

Example 29.4.2

For which states does the equation $_tp_x^{ii} = {}_tp_x^{\overline{ii}}$ hold?

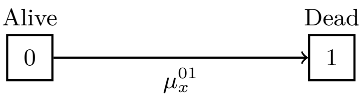


Solution 29.4.2

Only the "dead" state satisfies the equation and $_tp_x^{22} = {}_tp_x^{\overline{22}}$. For the other states, $_tp_x^{00} > {}_tp_x^{\overline{00}}$ and $_tp_x^{11} > {}_tp_x^{\overline{11}}$.

Example 29.4.3

For which states does the equation $_tp_x^{ii} = {}_tp_x^{\overline{ii}}$ hold?



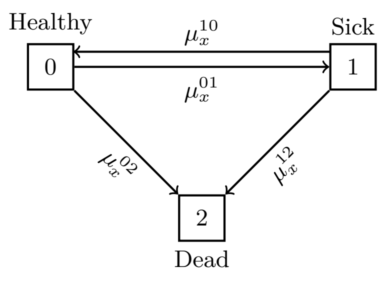
Solution 29.4.3

Both the "alive" and the "dead" states satisfy the equation $_tp_x^{ii} = {}_tp_x^{\overline{ii}}$.

29.5 KFE (Kolmogorov's forward equation)

KFE is $\frac{d}{dt}{}_tp_x^{ij} = \sum_{k=0,k\neq j}^{n} \left({}_tp_x^{ik}\,\mu_{x+t}^{kj} - {}_tp_x^{ij}\,\mu_{x+t}^{jk}\right)$. In a continuous multiple state model, except in simple cases such as $_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j=0;j\neq i}^{n} \mu_{x+s}^{ij}ds\right)$, often there are no easy formulas for $_tp_x^{ij}$ and we have to solve multiple KFEs. Before worrying about how to solve KFEs, let's first focus on how to correctly write KFEs.

Example 29.5.1



Write the formula for $\frac{d}{dt}{}_tp_x^{01}$.

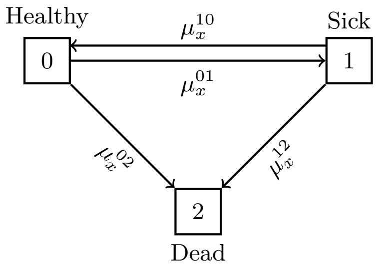
Solution 29.5.1

$\frac{d}{dt}{}_tp_x^{01} = \sum_{k=0,2}\left({}_tp_x^{0k}\mu_{x+t}^{k1} - {}_tp_x^{01}\mu_{x+t}^{1k}\right)$. To write the plus part, ask *"How can I start from the beginning state, go to a non-destination state, and finally get to the destination state?"* The only path is $0 \to 0 \to 1$. So $+{}_tp_x^{00}\mu_{x+t}^{01}$; plus because we are moving in. To write the minus part, ask *"How can I start from the beginning state, go to the destination state, and then get out of the destination state?"* Two paths: $0 \to 1 \to 0$ and $0 \to 1 \to 2$. Hence $-({}_tp_x^{01}\mu_{x+t}^{10} + {}_tp_x^{01}\mu_{x+t}^{12})$; minus because we are moving out.

$\frac{d}{dt}{}_tp_x^{01} = {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}(\mu_{x+t}^{10} + \mu_{x+t}^{12})$

Example 29.5.2

Write the formula for $\frac{d}{dt}{}_tp_x^{00}$.



Solution 29.5.2

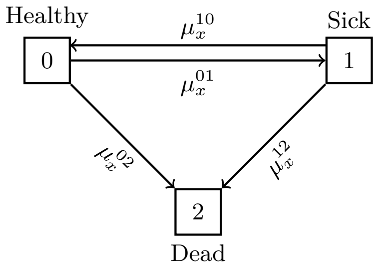
"How can I start from the beginning state, go to a non-destination state, and finally get to the destination state?" $0 \to 1 \to 0$.

"How can I start from the beginning state, go to the destination state, and then get out of the destination state?" $0 \to 0 \to 1$ and $0 \to 0 \to 2$.

$\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}(\mu_{x+t}^{01} + \mu_{x+t}^{02})$

Example 29.5.3

Write the formula for $\frac{d}{dt}{}_tp_x^{22}$.



Solution 29.5.3

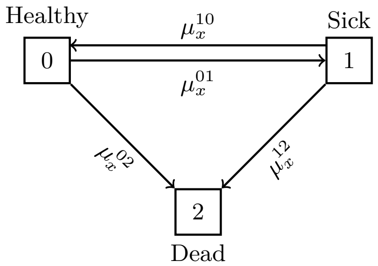
"How can I start from the beginning state, go to a non-destination state, and finally get to the destination state?" No way.

"How can I start from the beginning state, go to the destination state, and then get out of the destination state?" No way.

$\frac{d}{dt}{}_tp_x^{22} = 0$

Example 29.5.4

Write the formula for $\frac{d}{dt}{}_tp_x^{10}$, $\frac{d}{dt}{}_tp_x^{12}$, $\frac{d}{dt}{}_tp_x^{02}$.



Solution 29.5.4

$\frac{d}{dt}{}_tp_x^{10} = {}_tp_x^{11}\mu_{x+t}^{10} - {}_tp_x^{10}(\mu_{x+t}^{01} + \mu_{x+t}^{02})$

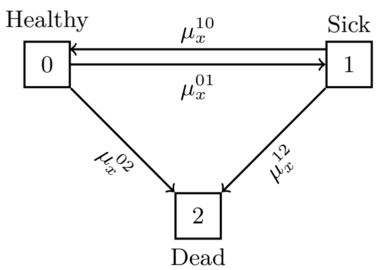
$\frac{d}{dt}{}_tp_x^{12} = {}_tp_x^{11}\mu_{x+t}^{12} + {}_tp_x^{10}\mu_{x+t}^{02}$

$\frac{d}{dt}{}_tp_x^{02} = {}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12}$

Example 29.5.5

Use KFE to derive the formula:

${}_tp_x^{\overline{00}} = \exp\left(-\int_0^t(\mu_{x+s}^{01} + \mu_{x+s}^{02})ds\right)$.



Solution 29.5.5

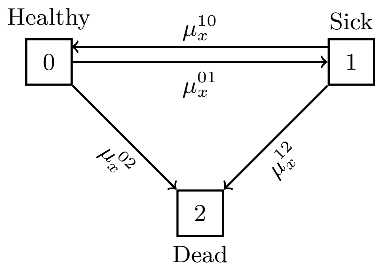
The KFE for ${}_tp_x^{00}$ is $\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}(\mu_{x+t}^{01} + \mu_{x+t}^{02})$. Since we want the state to be always 0, set ${}_tp_x^{01} = 0$ and we'll get the KFE for ${}_tp_x^{\overline{00}}$: $\frac{d}{dt}{}_tp_x^{\overline{00}} = -{}_tp_x^{\overline{00}}(\mu_{x+t}^{01} + \mu_{x+t}^{02})$, $\frac{1}{{}_tp_x^{\overline{00}}}\frac{d}{dt}{}_tp_x^{\overline{00}} = \frac{d}{dt}\ln {}_tp_x^{\overline{00}} = -(\mu_{x+t}^{01} + \mu_{x+t}^{02})$. Integrate both sides: ${}_tp_x^{\overline{00}} = C\exp\left(-\int_0^t(\mu_{x+s}^{01} + \mu_{x+s}^{02})ds\right)$, where C is a constant. Since ${}_0p_x^{\overline{00}} = 1$, $C = 1$ and ${}_tp_x^{\overline{00}} = \exp\left(-\int_0^t(\mu_{x+s}^{01} + \mu_{x+s}^{02})ds\right)$.

29.6 Euler method for solving KFEs

Example 29.6.1

You are given the following model (from AMLCR textbook Example 8.5)

- $\mu_x^{01} = a_1 + b_1 \exp(c_1 x)$, $\mu_x^{10} = 0.1\mu_x^{01}$, $\mu_x^{02} = a_2 + b_2 \exp(c_2 x)$, $\mu_x^{12} = \mu_x^{02}$
- $a_1 = 0.0004$, $b_1 = 3.4674 \times 10^{-6}$, $c_1 = 0.138155$, $a_2 = 0.0005$, $b_2 = 7.5858 \times 10^{-5}$, $c_2 = 0.087498$



Let $h = 1/12$ and $x = 60$. Calculate the following probabilities:

- ${}_hp_x^{00}$, the probability that a healthy (x) is still healthy one month from today
- ${}_hp_x^{01}$, the probability that a healthy (x) is sick one month from today
- ${}_hp_x^{02}$, the probability that a healthy (x) is dead one month from today
- ${}_{2h}p_x^{00}$, the probability that a healthy (x) is still healthy two months from today
- ${}_{2h}p_x^{01}$, the probability that a healthy (x) is sick two months from today
- ${}_{2h}p_x^{02}$, the probability that a healthy (x) is dead two months from today

Solution 29.6.1

The derivative of ${}_tp_x^{00}$ is $\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}(\mu_{x+t}^{01} + \mu_{x+t}^{02})$. Notice that the term ${}_tp_x^{00}$ appears on both sides. In addition, the righthand side has a term ${}_tp_x^{01}$. Like most other first order differential equations, this equation doesn't have an exact solution. However, we can use numerical methods to approximate solutions to differential equations. There are many methods to approximate solutions to a differential equation. One of the oldest and easiest method was originally devised by Euler and is called the Euler method.

This is the essence of the Euler method. Suppose we need to find the value of an unknown function $f(x)$ at $x = b$. We know the function's initial value $f(a)$. We also know the slope of $f(x)$ at any point. Then we can divide $[a, b]$ into n subintervals each of length $h = (b - a)/n$ and successively use the tangent line approximation to find $f(b)$. First, $f(a+h) \approx f(a) + hf'(a)$. Next, $f(a+2h) \approx f(a+h) + hf'(a+h)$. This process continues till $f(b) \approx f(b-h) + hf'(b-h)$. Now let's see the Euler method in action.

$${}_hp_x^{00} \approx {}_0p_x^{00} + h\Big[{}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}(\mu_{x+t}^{01} + \mu_{x+t}^{02})\Big]_{t=0} = {}_0p_x^{00} + h\Big[{}_0p_x^{01}\mu_x^{10} - {}_0p_x^{00}(\mu_x^{01} + \mu_x^{02})\Big]$$

$${}_0p_x^{00} = 1, \quad {}_0p_x^{01} = 0$$

$$\mu_x^{01} + \mu_x^{02} = a_1 + b_1 \exp(c_1 x) + a_2 + b_2 \exp(c_2 x)$$

$$= 0.0004 + 3.4674 \times 10^{-6} e^{0.138155(60)} + 0.0005 + 7.5858 \times 10^{-5} e^{0.087498(60)} = 0.029158122$$

$$\Rightarrow {}_hp_x^{00} \approx 1 - \frac{1}{12}(\mu_x^{01} + \mu_x^{02}) = 1 - \frac{0.029158122}{12} = 0.997570$$

$$\frac{d}{dt}{}_tp_x^{01} = {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}(\mu_{x+t}^{10} + \mu_{x+t}^{12})$$

$$\Rightarrow {}_hp_x^{01} \approx {}_0p_x^{01} + h\Big[{}_0p_x^{00}\mu_x^{01} - {}_0p_x^{01}(\mu_x^{10} + \mu_x^{12})\Big] = h\mu_x^{01}$$

$$= \frac{0.0004 + 3.4674 \times 10^{-6} e^{0.138155(60)}}{12} = 0.001184$$

$$\frac{d}{dt}{}_tp_x^{02} = {}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12}$$

$${}_hp_x^{02} \approx {}_0p_x^{02} + h\Big[{}_0p_x^{00}\mu_x^{02} + {}_0p_x^{01}\mu_x^{12}\Big] = h\mu_x^{02}$$

$$= \frac{0.014954241}{12} = 0.001246$$

Alternatively,

$${}_hp_x^{02} = 1 - ({}_hp_x^{00} + {}_hp_x^{01}) = 1 - (0.997570 + 0.001184) = 0.001246$$

$${}_{2h}p_x^{00} \approx {}_hp_x^{00} + h\Big[{}_hp_x^{01}\mu_{x+h}^{10} - {}_hp_x^{00}(\mu_{x+h}^{01} + \mu_{x+h}^{02})\Big]$$

$$= 0.997570 + \frac{0.001184 \times 0.001436372 - 0.997570(0.014363722 + 0.01506002)}{12} = 0.995124$$

$${}_{2h}p_x^{01} \approx {}_hp_x^{01} + h\Big[{}_hp_x^{00}\mu_{x+h}^{01} - {}_hp_x^{01}(\mu_{x+h}^{10} + \mu_{x+h}^{12})\Big]$$

$$= 0.001184 + \frac{0.997570 \times 0.014363722 - 0.001184(0.001436372 + 0.01506002)}{12} = 0.002376$$

$${}_{2h}p_x^{02} \approx {}_hp_x^{02} + h\left[{}_hp_x^{00}\mu_{x+h}^{02} + {}_hp_x^{01}\mu_{x+h}^{12}\right]$$

$$= 0.001246 + \frac{0.997570 \times 0.014363722 + 0.001184 \times 0.01506002}{12} = 0.002500$$

Alternatively,

$${}_{2h}p_x^{02} = 1 - \left({}_{2h}p_x^{00} + {}_{2h}p_x^{01}\right) = 1 - (0.995124 + 0.002376) = 0.002500$$

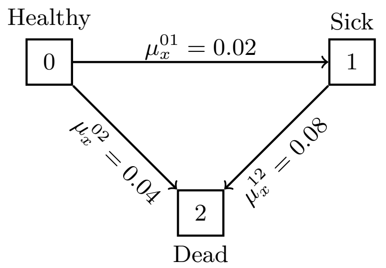
By the way, the Euler method does not require you to divide the interval $[a, b]$ into subintervals of an equal length. However, equal length subintervals are often chosen for the ease of implementation.

29.7 getting back and being stuck, various transition probabilities

In a word problem, it may not be immediately clear whether you should use ${}_tp_x^{ii}$ or ${}_tp_x^{\overline{ii}}$. If re-entry to state i is impossible, then ${}_tp_x^{ii} = {}_tp_x^{\overline{ii}}$ and it doesn't matter which one you use. However, if re-entry is possible, then ${}_tp_x^{ii} \neq {}_tp_x^{\overline{ii}}$ and water gets muddy. Ask "Is re-entry to state i allowed in the event?" If YES, use ${}_tp_x^{ii}$. If NO, then use ${}_tp_x^{\overline{ii}}$.

Example 29.7.1

A 10-year sickness policy on a healthy life (50) pays 100,000 at the moment when the insured becomes sick. $\delta = 0.06$. Calculate the EPV of this policy.



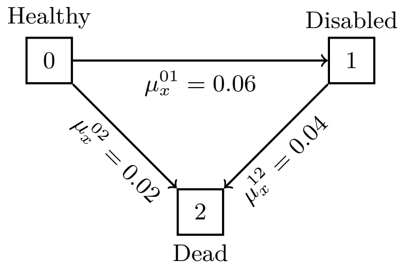
Solution 29.7.1

The death benefit is paid at t if the insured (1) is still in state 0 at t (e.g. neither dead nor disabled at t), prob: ${}_tp_{50}^{00}$, and (2) transitions to state 1 during $[t, t + dt]$, prob: $\mu_{50+t}^{01}dt$.

$100000 \int_0^{10} e^{-\delta t}{}_tp_{50}^{00}\mu_{50+t}^{01}dt = 100000 \int_0^{10} e^{-\delta t}{}_tp_{50}^{\overline{00}}\mu_{50+t}^{01}dt = 100000 \int_0^{10} e^{-0.06t}e^{-(0.02+0.04)t}0.02dt = \frac{100000 \times 2}{12}\left(1 - e^{-0.12(10)}\right) = 11646.763$

Example 29.7.2

The transition intensities are constants for all ages. Calculate the probability that a healthy life (x) today is still healthy 10 years from today.

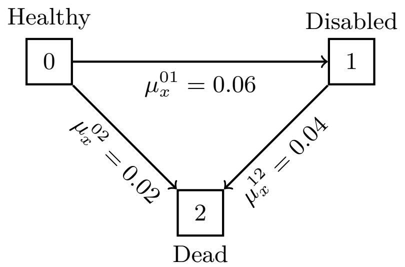


Solution 29.7.2

${}_{10}p_x^{00} = {}_{10}p_x^{\overline{00}} = e^{-(0.06+0.02)10} = e^{-0.8}$.

Example 29.7.3

The transition intensities are constants for all ages. Calculate the probability that a healthy life (x) today is disabled 10 years from today.

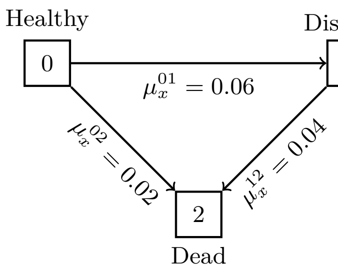


Solution 29.7.3

We want to start from state 0 today and arrive at state 1 ten years later. We can (1) hang out at state 0 during $[0, t]$, prob: ${}_tp_x^{\overline{00}} = e^{-0.08t}$, (2) instantly transition to state 1 during $[t, t + dt]$, prob: ${}_{dt}p_{x+t}^{01} = \mu_{x+t}^{01}dt + o(dt) \approx 0.06dt$, and (3) hang out in state 1 during $[t+dt, 10] \approx [t, 10]$, prob: ${}_{10-t}p_{x+t}^{\overline{11}} = e^{-0.04(10-t)}$. For any $0 \leq t \leq 10$, the total probability of the 3 parts is $p(t) = e^{-0.08t}0.06dte^{-0.04(10-t)}$. We sum $p(t)$ by all the t's from 0 to 10: ${}_{10}p_x^{01} = \int_0^{10} {}_tp_x^{\overline{00}}\mu_{x+t}^{01}{}_{10-t}p_{x+t}^{\overline{11}}dt = \int_0^{10} e^{-0.08t}0.06e^{-0.04(10-t)}dt = 0.06e^{-0.4}\int_0^{10} e^{-0.04t}dt = \frac{6e^{-0.4}}{4}\left(1 - e^{-0.4}\right) = 0.3315$

Example 29.7.4

The transition intensities are constants for all ages. A healthy insured is age x today. Let A represent the probability that the insured is dead 10 years from today and he's disabled before death. Calculate A.



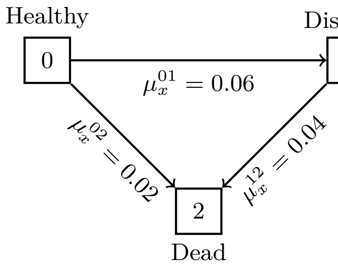
Solution 29.7.4

We want to complete the path $0 \to 1 \to 2$ in 10 years. $A = \int_0^{10} {}_tp_x^{01}\mu_{x+t}^{12}\,{}_{10-t}p_{x+t}^{22}dt = \int_0^{10} {}_tp_x^{01}\mu_{x+t}^{12}dt$. From the previous problem, ${}_mp_x^{01} = \int_0^m {}_tp_x^{\overline{00}}\mu_{x+t}^{01}\,{}_{m-t}p_{x+t}^{\overline{11}}dt = \int_0^m e^{-0.08t}0.06e^{-0.04(m-t)}dt = 0.06e^{-0.04m}\int_0^m e^{-0.04t}dt = \frac{0.06}{0.04}\left(e^{-0.04m} - e^{-0.08m}\right)$.

$A = \int_0^{10} {}_tp_x^{01}\mu_{x+t}^{12}dt = \int_0^{10} \frac{0.06}{0.04}\left(e^{-0.04t} - e^{-0.08t}\right)0.08dt = 0.12\left(\frac{1-e^{-0.4}}{0.04} - \frac{1-e^{-0.8}}{0.08}\right) = 0.16303$

Example 29.7.5

The transition intensities are constants for all ages. A healthy insured is age x today. Let B represent the probability that the insured is dead 10 years from today and he's healthy before death. Calculate B.

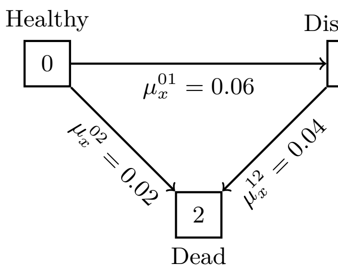


Solution 29.7.5

We want to complete the path $0 \to 2$ in 10 years. $B = \int_0^{10} {}_tp_x^{00}\mu_{x+t}^{02}dt = \int_0^{10} {}_tp_x^{\overline{00}}\mu_{x+t}^{02}dt = \int_0^{10} e^{-0.08t}0.02dt = 0.02\frac{1-e^{-0.8}}{0.08} = 0.1377$

Example 29.7.6

The transition intensities are constants for all ages. A healthy insured is age x today. Let C represent the probability that the insured is dead 10 years from today.

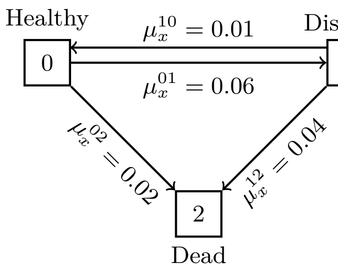


Solution 29.7.6

$C = A + B = 0.16303 + 0.1377 = 0.30073$

Example 29.7.7

The transition intensities are constants for all ages. Calculate the probability that a healthy life (x) today is still healthy 10 years from today. Is it ${}_{10}p_x^{00}$ or ${}_{10}p_x^{\overline{00}}$?



Solution 29.7.7

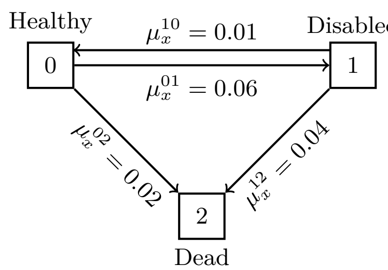
Now ${}_{10}p_x^{00} > {}_{10}p_x^{\overline{00}}$. Which one to use?

When you count the number of people still healthy at $t = 10$, should you include those who return to the healthy state after recovering from prior disabilities? Yes you should. Then the probability is ${}_{10}p_x^{00}$.

There's no closed-form formula for ${}_{10}p_x^{00}$. We can use the Euler method to approximate ${}_{10}p_x^{00}$.

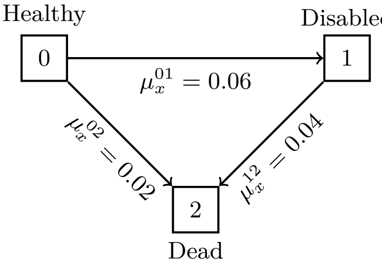
Example 29.7.8

The transition intensities are constants for all ages. Calculate the probability that a healthy life (x) today is ever disabled during the next 10 years.



Solution 29.7.8

The insured can travel back and forth between state 0 and state 1 repeatedly and have many disability relapses. Do disability relapses matter in this problem? Surprisingly NO. The event "ever being disabled" is the same as walking through the path $0 \to 1$ at least once, which is the same as having the 1st period of disability. We can simplify the diagram into:



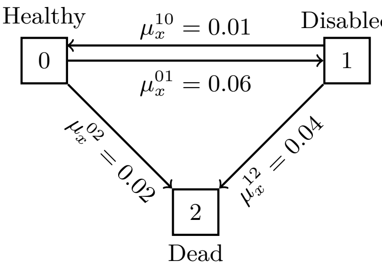
The insured can (1) hang out at state 0 during $[0, t]$, prob: ${}_tp_x^{\overline{00}}$, and (2) move from state 0 to 1 in the next instant, prob: $\mu_{x+t}^{01}dt$. The probability of making these two moves is $p(t) = {}_tp_x^{\overline{00}}\mu_{x+t}^{01}dt$. Next, sum $p(t)$ from $t = 0$ to $t = 10$: $\int_0^{10} {}_tp_x^{\overline{00}}\mu_{x+t}^{01}dt = \int_0^{10} e^{-0.08t}0.06dt = \frac{6}{8}\left(1 - e^{-0.8}\right) = 0.4130$.

Example 29.7.9

The transition intensities are constants for all ages. Consider two probabilities:

- A, as calculated in the last problem, is the probability that a healthy life (x) today is disabled at some point during the next 10 years.
- $B = {}_{10}p_x^{01}$ is the probability that a healthy life (x) today is disabled at the end of Year 10.

Actuary Moray is puzzled by the fact that there's an exact solution to A yet we have to use the Euler method to approximate B. Help Moray understand why.



Solution 29.7.9

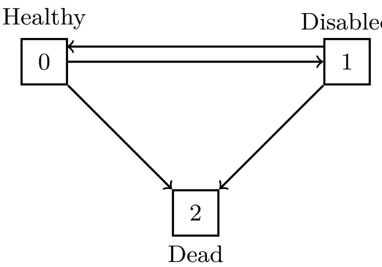
A is the probability that the *first* disability occurs during the next 10 years. In B, however, the insured can travel back and forth between state 0 and 1 any number of times (e.g. becoming newly disabled again and again) as long as he's disabled at the end of Year 10. B is the probability that a healthy life now becomes newly disabled at the end of Year 10 and that this disability is the n-th time (where $n = 1, 2, \ldots$) that the insured is disabled during the first 10 years. Clearly, B is much harder to find.

Example 29.7.10

Which expression is the probability that a healthy life (x) today is disabled during the first 10 years and remains disabled throughout the remainder of the first 10 years?

(A) $\int_0^{10} {}_tp_x^{\overline{00}}\mu_{x+t}^{01}{}_{10-t}p_{x+t}^{\overline{11}}dt$

(B) $\int_0^{10} {}_tp_x^{01}{}_{10-t}p_{x+t}^{\overline{11}}dt$



Solution 29.7.10

$\int_0^{10} {}_tp_x^{\overline{00}}\mu_{x+t}^{01}{}_{10-t}p_x^{\overline{11}}dt$ is the probability that the insured's *first* disability lasts throughout the remainder of the first 10 years, whereas $\int_0^{10} {}_tp_x^{01}{}_{10-t}p_{x+t}^{\overline{11}}dt$ is the probability that the insured's *any* disability lasts throughout the remainder of the first 10 years. The correct expression is $\int_0^{10} {}_tp_x^{01}{}_{10-t}p_{x+t}^{\overline{11}}dt$.

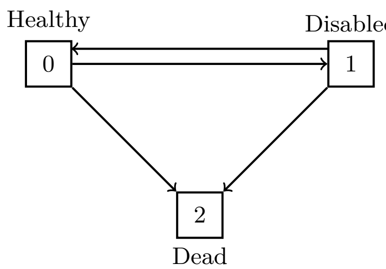
Theoretically, for example, an insured can have many "healthy this month, disabled next month" cycles before finally becoming disabled continuously throughout the remainder of the first 10 years. Such a scenario is discarded in $\int_0^{10} {}_tp_x^{\overline{00}}\mu_{x+t}^{01}{}_{10-t}p_x^{\overline{11}}dt$ but is captured in $\int_0^{10} {}_tp_x^{01}{}_{10-t}p_{x+t}^{\overline{11}}dt$.

Example 29.7.11

A 10-year disability insurance is issued to a healthy life (x). Premiums are payable continuously at the rate of P per year while the insured is healthy. Which expression is the EPV of the premiums?

(A) $P\int_0^{10} e^{-\delta t}{}_tp_x^{\overline{00}}dt$

(B) $P\int_0^{10} e^{-\delta t}{}_tp_x^{00}dt$

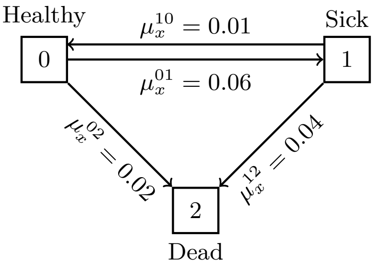


Solution 29.7.11

If the insured returns to the healthy state after recovering from disability during the term of the policy, will the insured pay the premium? YES. The EPV is $P\int_0^{10} e^{-\delta t}\,{}_tp_x^{00}dt$.

Example 29.7.12

A 10-year sickness insurance policy on a healthy life (x) pays a first-sickness-recovery bonus. If the insured is sick but later recovers from his first sickness during the term of the contract, a bonus of 100 is immediately paid upon recovery. The transition intensities are constants for all ages. $\delta = 0.07$. Calculate the EPV of the bonus.



Solution 29.7.12

The bonus is paid if the insured walks through the path $0 \to 1 \to 0$ in 10 years. The insured needs to do the following: (1) stay in state 0 during $[0, t]$, prob: ${}_tp_x^{\overline{00}}$, (2) transition to state 1 during $[t, t+dt]$, prob: $\mu_{x+t}^{01}dt$, (3) stay in state 1 during $[t, t+u]$, prob: ${}_up_{x+t}^{\overline{11}}$, and (4) finally transition to state 0 during $[t+u, t+u+du]$, prob: $\mu_{x+t+u}^{10}du$. The bonus is paid at $t+u$. The constraints are $t \geq 0$, $u \geq 0$, and $0 \leq t+u \leq 10$.

$100\int_{t=0}^{10}\int_{u=0}^{10-t} e^{-\delta(t+u)}\,{}_tp_x^{\overline{00}}\mu_{x+t}^{01}\,{}_up_{x+t}^{\overline{11}}\mu_{x+t+u}^{10}dudt = 100\int_{t=0}^{10}\int_{u=0}^{10-t} e^{-0.07(t+u)}e^{-0.08t}0.06e^{-0.05u}0.01dudt = 0.06\int_0^{10} e^{-0.15t}\int_0^{10-t} e^{-0.12u}dudt = 0.06\int_0^{10}\frac{1-e^{-0.12(10-t)}}{0.12}e^{-0.15t}dt$

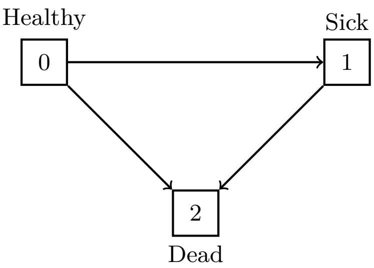
$= 0.5\left(\int_0^{10} e^{-0.15t}dt - e^{-1.2}\int_0^{10} e^{-0.03t}dt\right)$

$$= 0.5\left(\frac{1-e^{-1.5}}{0.15} - e^{-1.2}\frac{1-e^{-0.3}}{0.03}\right) = 1.2885$$

29.8 Check your knowledge

Homework 29.8.1

(MLC: Spring 2016 Q4) A 5-year sickness insurance policy is based on the following Markov model:



You are given the following constant forces of transition:

(i) $\mu^{01} = 0.05$

(ii) $\mu^{10} = 0.02$

(iii) $\mu^{02} = 0.01$

(iv) $\mu^{12} = 0.06$

Calculate the probability that a Healthy life will become Sick exactly once during the 5 years and remain continuously Sick from that point until the end of the 5 years.

Homework Solution 29.8.1

★★★★☆ Difficulty

$\int_0^5 {}_tp_x^{\overline{00}}\mu_{x+t}^{01}\,{}_{5-t}p_{x+t}^{\overline{11}}dt = \int_0^5 e^{-(0.05+0.01)t}0.05e^{-(5-t)(0.02+0.06)}dt = 0.17624544$

Homework 29.8.2

(spring 2012 MLC Q12) Employees in Company ABC can be in:

State 0: Non-executive employee

State 1: Executive employee

State 2: Terminated from employment

John joins Company ABC as a non-executive employee at age 30. You are given:

(i) $\mu^{01} = 0.01$ for all years of service

(ii) $\mu^{02} = 0.06$ for all years of service

(iii) $\mu^{12} = 0.02$ for all years of service

(iv) Executive employees never return to the non-executive employee state.

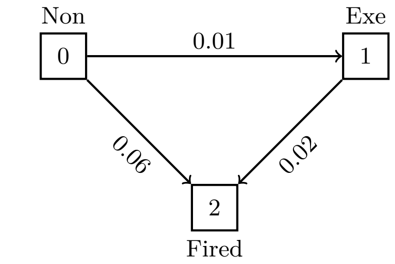
(v) Employees terminated from employment never get re-hired.

(vi) The probability that John lives to age 65 is 0.9, regardless of state.

Calculate the probability that John will be an executive employee of Company ABC at age 65.

Homework Solution 29.8.2

★★★★☆ Difficulty



The probability that John will be an executive employee of Company ABC at age 65 is:

${}_{35}p_{30} \times {}_{35}p_{30}^{01} = 0.9\,{}_{35}p_{30}^{01}$

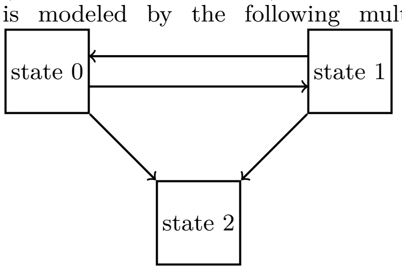
${}_{35}p_{30}^{01} = \int_0^{35} {}_tp_{30}^{\overline{00}}\mu_{30+t}^{01}\,{}_{35-t}p_{30+t}^{\overline{11}}dt$

$= \int_0^{35} e^{-(0.01+0.06)t}0.01e^{-0.02(35-t)}dt = 0.258$

The desired probability is $0.9(0.258) = 0.2322$

Homework 29.8.3

(MLC: Spring 2014 Q3) A continuous Markov process is modeled by the following multiple state diagram:



You are given the following constant transition intensities:

(i) $\mu^{01} = 0.08$

(ii) $\mu^{02} = 0.04$

(iii) $\mu^{10} = 0.10$

(iv) $\mu^{12} = 0.05$

For a person in State 1, calculate the probability that the person will continuously remain in State 1 for the next 15 years.

Homework Solution 29.8.3

★★★☆☆ Difficulty

${}_{15}p_x^{\overline{11}} = \exp\left(-\int_0^{15}(\mu_{x+s}^{10} + \mu_{x+s}^{12})ds\right) = e^{-15(0.15)} = 0.1054$

Homework 29.8.4

(Exam MLC: Spring 2012 Q28) You are using Euler's method to calculate estimates of probabilities for a multiple state model with states $0, 1, 2$. You are given:

(i) The only possible transitions between states are: 0 to 1, 1 to 0, and 1 to 2

(ii) For all x, $\mu_x^{01} = 0.3$, $\mu_x^{10} = 0.1$, $\mu_x^{12} = 0.1$

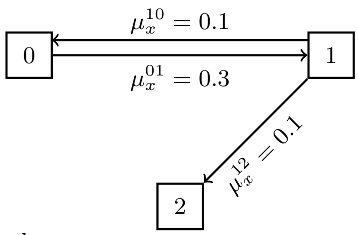
(iii) Your step size is 0.1.

(iv) You have calculated that ${}_{0.6}p_x^{00} = 0.8370$, ${}_{0.6}p_x^{01} = 0.1588$, ${}_{0.6}p_x^{02} = 0.0042$,

Calculate the estimate of ${}_{0.8}p_x^{01}$ using the specified procedure.

Homework Solution 29.8.4

★★★☆☆ Difficulty



$\frac{d}{dt}{}_tp_x^{00} = t\left[{}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}\mu_{x+t}^{01}\right] = t\left[0.1{}_tp_x^{01} - 0.3{}_tp_x^{00}\right]$

$\frac{d}{dt}{}_tp_x^{01} = t\left[{}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}(\mu_{x+t}^{10} + \mu_{x+t}^{12})\right] = t\left[0.3{}_tp_x^{00} - 0.2{}_tp_x^{01}\right]$

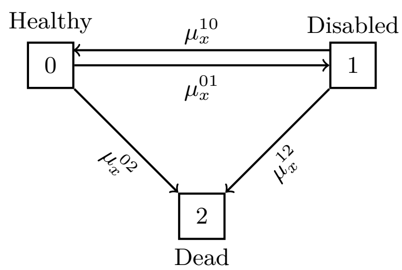
${}_{0.7}p_x^{00} \approx {}_{0.6}p_x^{00} + 0.1(0.1{}_{0.6}p_x^{01} - 0.3{}_{0.6}p_x^{00}) = 0.8370 + 0.1(0.1(0.1588) - 0.3(0.8370)) = 0.813478$

${}_{0.7}p_x^{01} \approx {}_{0.6}p_x^{01} + 0.1(0.3{}_{0.6}p_x^{00} - 0.2{}_{0.6}p_x^{01}) = 0.1588 + 0.1(0.3(0.8370) - 0.2(0.1588)) = 0.180734$

${}_{0.8}p_x^{01} \approx {}_{0.7}p_x^{01} + 0.1(0.3{}_{0.7}p_x^{00} - 0.2{}_{0.7}p_x^{01}) = 0.180734 + 0.1(0.3(0.813478) - 0.2(0.180734)) = 0.20152366$

Homework 29.8.5

The transition intensities are constants for all ages. Write down the KFEs for the following model:



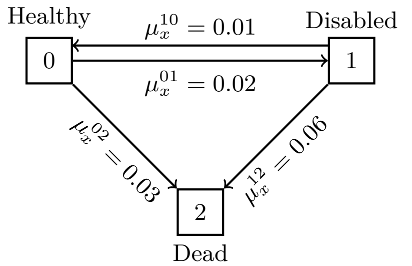
Homework Solution 29.8.5

★★★★☆ Difficulty

$$\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}(\mu_{x+t}^{01} + \mu_{x+t}^{02})$$
$$\frac{d}{dt}{}_tp_x^{01} = {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}(\mu_{x+t}^{10} + \mu_{x+t}^{12})$$
$$\frac{d}{dt}{}_tp_x^{02} = {}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12}$$
$$\frac{d}{dt}{}_tp_x^{11} = {}_tp_x^{10}\mu_{x+t}^{01} - {}_tp_x^{11}(\mu_{x+t}^{10} + \mu_{x+t}^{12})$$
$$\frac{d}{dt}{}_tp_x^{10} = {}_tp_x^{11}\mu_{x+t}^{10} - {}_tp_x^{10}(\mu_{x+t}^{01} + \mu_{x+t}^{02})$$
$$\frac{d}{dt}{}_tp_x^{12} = {}_tp_x^{11}\mu_{x+t}^{12} + {}_tp_x^{10}\mu_{x+t}^{02}$$
$${}_tp_x^{22} = 1 \text{ and } \frac{d}{dt}{}_tp_x^{22} = 0$$

Homework 29.8.6

A disability insurance is issued to healthy life (x). The transition intensities are constants for all ages. Let the contract issue time be time zero. Use the Euler method. Set $h = 1/12$. Approximate ${}_{3h}p_x^{00}$, ${}_{3h}p_x^{01}$, ${}_{3h}p_x^{11}$, ${}_{3h}p_x^{10}$. In addition, estimate ${}_{3h}p_x^{00} - {}_{3h}p_x^{\overline{00}}$.



Homework Solution 29.8.6

★★★★☆ Difficulty

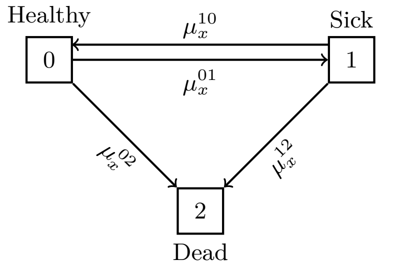
t	${}_tp_x^{00}$	${}_tp_x^{01}$	${}_tp_x^{02}$
0	1.00000	0.00000	0.00000
h	0.99583	0.00167	0.00250
$2h$	0.99169	0.00332	0.00500
$3h$	0.98756	0.00495	0.00749

$${}_{3h}p_x^{\overline{00}} = \int_0^{3/12} e^{-0.05t}dt = \frac{1 - e^{-0.05\times 3/12}}{0.05} = 0.24844,$$
$${}_{3h}p_x^{00} - {}_{3h}p_x^{\overline{00}} = 0.98756 - 0.24844 = 0.73912$$

Homework 29.8.7

You are applying the Euler method to the following model (AMLCR textbook Example 8.5):

- $\mu_x^{01} = a_1 + b_1\exp(c_1x)$, $\mu_x^{10} = 0.1\mu_x^{01}$, $\mu_x^{02} = a_2 + b_2\exp(c_2x)$, $\mu_x^{12} = \mu_x^{02}$
- $a_1 = 0.0004$, $b_1 = 3.4674 \times 10^{-6}$, $c_1 = 0.138155$, $a_2 = 0.0005$, $b_2 = 7.5858 \times 10^{-5}$, $c_2 = 0.087498$



Let $h = 1/12$ and $x = 50$. For $n = 96$, ${}_{nh}p_x^{00} = 0.880641$ and ${}_{nh}p_x^{01} = 0.048416$. Calculate ${}_{(n+1)h}p_x^{00}$.

Homework Solution 29.8.7

★★★★☆ Difficulty

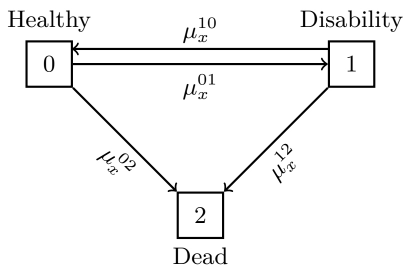
$$\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}(\mu_{x+t}^{01} + \mu_{x+t}^{02}),$$
$${}_{(n+1)h}p_x^{00} \approx {}_{nh}p_x^{00} + h\left({}_{nh}p_x^{01}\mu_{x+nh}^{10} - {}_{nh}p_x^{00}(\mu_{x+nh}^{01} + \mu_{x+nh}^{02})\right)$$

$nh = 96/12 = 8$, $\mu_{58}^{01} = 0.010871316$, $\mu_{58}^{02} = 0.012633763$, $\mu_{58}^{10} = 0.001087132$

$${}_{(n+1)h}p_x^{00} \approx 0.880641 + (1/12)\Big(0.048416(0.001087132) - 0.880641(0.010871316 + 0.012633763)\Big) = 0.87892$$

Homework 29.8.8

A 10-year disability insurance policy is issued to healthy life (x). The policy pays 100,000 immediately at the onset of disability.



Which expression is the EPV for this policy?

(A) $100000\int_0^{10} e^{-\delta t}{}_tp_x^{00}\mu_{x+t}^{01}dt$

(B) $100000\int_0^{10} e^{-\delta t}{}_tp_x^{\overline{00}}\mu_{x+t}^{01}dt$

Homework Solution 29.8.8

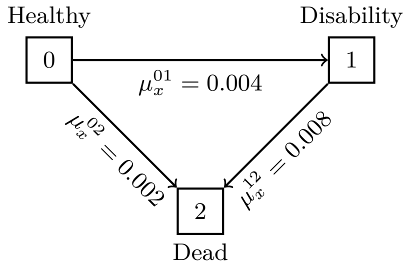
★★★★☆ Difficulty

A is correct. The insured can be newly disabled many times during the term of the contract and each new disability triggers the benefit. A counts for this while B is the EPV of the benefit for the first disability during the term of the contract. There's no exact way to calculate the integral $100000\int_0^{10} e^{-\delta t}{}_tp_x^{00}\mu_{x+t}^{01}dt$ as there's no exact way to find ${}_tp_x^{00}$. You can use the trapezoidal rule or the Simpson's rule or other numerical integration for approximation. The Simpson's rule is $\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$.

Homework 29.8.9

A combined 10-year term life and disability insurance policy is issued to healthy life (x). The policy pays 100,000 immediately on death or the onset of disability. No further benefit is paid in the event of death after a prior disability claim has been paid.

The transition intensities are constants for all ages. $\delta = 0.05$. Calculate the EPV of this policy.



Homework Solution 29.8.9

★★★★☆ Difficulty

We can ignore path $1 \to 2$ because death after disability will not trigger the death benefit.

$100000 \int_0^{10} e^{-\delta t}\, {}_tp_x^{00}(\mu_{x+t}^{01} + \mu_{x+t}^{02})dt$

$= 100000 \int_0^{10} e^{-\delta t}\, {}_tp_x^{\overline{00}}(\mu_{x+t}^{01} + \mu_{x+t}^{02})dt$

$= 100000 \int_0^{10} e^{-0.05t} e^{-0.006t} 0.006dt = 100000 \int_0^{10} e^{-0.056t} 0.006dt$

$= 100000 \times \dfrac{6}{56}\left(1 - e^{-0.056\times 10}\right) = 4594.1886$

Homework 29.8.10

Same as the last problem EXCEPT that the limitation "no further benefit is paid in the event of death after a prior disability claim has been paid" is removed. Calculate the EPV of this policy.

Homework Solution 29.8.10

★★★★★ Difficulty

METHOD 1. EPV of the death benefit if the insured is healthy at death:

$$100000 \int_0^{10} e^{-\delta t}\, {}_tp_x^{00} \mu_{x+t}^{02} dt = 100000 \int_0^{10} e^{-0.05t} e^{-0.006t} 0.002dt = 100000 \times \frac{2}{56}\left(1 - e^{-0.56}\right) = 1531.3962$$

EPV of the disability benefit if the insured is continuously disabled throughout the remainder of the 10-year term:

$$100000 \int_0^{10} e^{-\delta t}\, {}_tp_x^{00} \mu_{x+t}^{01}\, {}_{10-t}p_{x+t}^{\overline{11}} dt = 100000 \int_0^{10} e^{-0.05t} e^{-0.006t} 0.004 e^{-(10-t)0.008} dt$$

$$= 100000(0.004)e^{-0.08} \int_0^{10} e^{-0.048t} dt = 100000(0.004)e^{-0.08} \frac{1 - e^{-0.48}}{0.048} = 2932.5607$$

The insured will get paid twice if he walks through the path $0 \to 1 \to 2$ during the 10-year term: (1) the disability benefit 100,000 at t, and (2) the death benefit 100,000 at $t + u$ if he makes the following moves:

(a) is continuously healthy during $[0, t]$; prob: ${}_tp_x^{\overline{00}} = e^{-0.006t}$

(b) becomes disabled during $[t, t + dt]$; prob: $\mu_{x+t}^{01} dt = 0.004dt$

(c) is continuously disabled during $[t + dt, t + dt + u] \approx [t, t + u]$; prob: ${}_up_{x+t}^{\overline{11}} = e^{-0.008u}$

(d) moves to state 2 during $[t + u, t + u + du]$, prob: $\mu_{x+t+u}^{12} du = 0.008du$.

For each (t, u) pair where $0 < t + u < 10$, the EPV of the double payments is

$$g(t, u) = 100,000(e^{-\delta t} + e^{-\delta(t+u)})e^{-0.006t} 0.004 e^{-0.008u} 0.008dudt$$

$$\int_{t=0}^{10} \int_{u=0}^{10-t} g(t, u) = 100,000 \int_{t=0}^{10} \int_{u=0}^{10-t} (e^{-0.05t} + e^{-0.05(t+u)})e^{-0.006t} 0.004 e^{-0.008u} 0.008dudt$$

$$= 100000(0.004)(0.008) \int_0^{10} e^{-0.056t} \left(\int_0^{10-t} (1 + e^{-0.05u})e^{-0.008u} du \right) dt = 240.6666$$

Total EPV: $1531.3962 + 2932.5607 + 240.6666 = 4704.6235$

If double payments are not allowed, the EPV for death after disability is

$$100,000 \int_{t=0}^{10} \int_{u=0}^{10-t} (e^{-0.05t} + 0)e^{-0.006t} 0.004 e^{-0.008u} 0.008dudt = 130.23171$$

Total EPV: $1531.3962 + 2932.5607 + 130.23171 = 4594.1886$

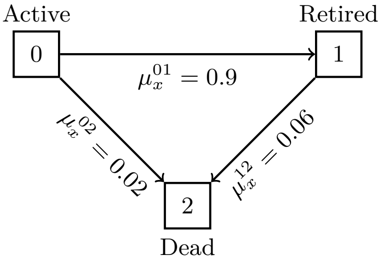
METHOD 2. EPV of the death benefit if the insured is disabled at death:

$$100000 \int_0^{10} \int_0^{10-t} e^{-\delta(t+u)}\, {}_tp_x^{00} \mu_{x+t}^{01}\, {}_up_{x+t}^{\overline{11}} \mu_{x+t+u}^{12} dudt = 100000 \int_0^{10} \int_0^{10-t} e^{-0.05(t+u)} e^{-0.006t} 0.004 e^{-0.008u} 0.008dudt = 110.4349$$

Total EPV: $4594.1886 + 110.4349 = 4704.6235$

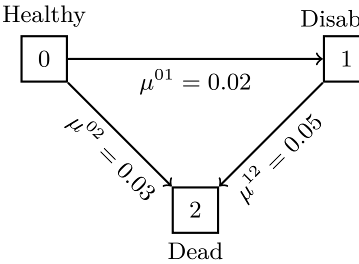
Homework 29.8.11

A pension plan provides a benefit of 100,000 payable on death regardless of whether death occurs before or after retirement. The transition intensities are constants for all ages. $\delta = 0.05$. Calculate the EPV of this policy for an active member currently age x.



Homework 29.8.12

(Exam MLC: Fall 2013 Q10) Calculate the conditional probability that a Healthy life on January 1, 2004 is still Healthy on January 1, 2014, given that this person is not Dead on January 1, 2014.



Homework 29.8.13

(Exam MLC: Fall 2013 Q21) You are pricing an automobile insurance on (x). The insurance pays 10,000 immediately if (x) gets into an accident within 5 years of issue. The policy pays only for the first accident and has no other benefits.

(i) You model (x)'s driving status as a multi-state model with the following 3 states:

0 - low risk, without an accident
1 - high risk, without an accident
2 - has had an accident

(ii) (x) is initially in state 0.

(iii) The following transition intensities for $0 \le t \le 5$

$\mu_{x+t}^{01} = 0.20 + 0.10t$
$\mu_{x+t}^{02} = 0.05 + 0.05t$
$\mu_{x+t}^{12} = 0.15 + 0.01t^2$

(iv) ${}_3p_x^{01} = 0.4174$

(v) $\delta = 0.02$

(vi) The continuous function $g(t)$ is such that the expected present value of the benefit up to time a equals $\int_0^a g(t)dt$, $0 \le a \le 5$, where t is the time of the first accident.

Calculate $g(3)$.

Homework 29.8.14

(Exam MLC: Fall 2012 Q12) A party of scientists arrives at a remote island. Unknown to them, a hungry tyrannosaur lives on the island. You model the future lifetimes of the scientists as a three-state model, where:

0 - State 0: no scientists have been eaten.

1 - State 1: exactly one scientist has been eaten.

2 - State 2: at least two scientists have been eaten.

Homework Solution 29.8.11

★★★★☆ Difficulty

EPV for the path $0 \to 2$: $100000 \int_0^\infty e^{-\delta t} {}_tp_x^{00} \mu_{x+t}^{02} dt = 100000 \int_0^\infty e^{-0.05t} e^{-0.92t} 0.02 dt = 2061.8557$

EPV for the path $0 \to 1 \to 2$ (the death benefit is paid at age $x+t+u$):

$100000 \int_0^\infty {}_tp_x^{\overline{00}} \mu_{x+t}^{01} \int_0^\infty e^{-\delta(t+u)} {}_up_{x+t}^{\overline{11}} \mu_{x+t+u}^{12} du dt$

$= 100000 \int_0^\infty e^{-0.92t} 0.9 \int_0^\infty e^{-0.05(t+u)} e^{-0.06u} 0.06 du dt$

$= 100000 \times 0.9 \times 0.06 \int_0^\infty e^{-0.97t} dt \int_0^\infty e^{-0.11u} du$

$= \dfrac{100000 \times 0.9 \times 0.06}{0.97 \times 0.11} = 50609.185$

Total: 2061.8557+50609.185=52671.041

Homework Solution 29.8.12

★★★☆☆ Difficulty

A =healthy today and healthy 10 years from today. B =healthy today, disabled at some point during the next 10 years and remain disabled during the reminder of the 10 year period.

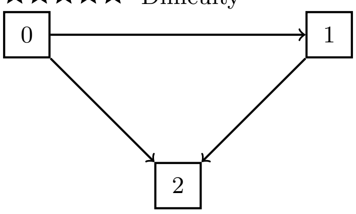
$P(A) = {}_{10}p_x^{00} = {}_{10}p_x^{\overline{00}} = e^{-(\mu^{01}+\mu^{02})10} = e^{-0.05 \times 10} = e^{-0.5}$

$P(B) = \int_0^{10} {}_tp_x^{\overline{00}} \mu^{01} {}_{10-t}p_{x+t}^{\overline{11}} dt = \int_0^{10} e^{-(\mu^{01}+\mu^{02})t} \mu^{01} e^{-\mu^{12}(10-t)} dt = \int_0^{10} e^{-0.05t} 0.02 e^{-0.05(10-t)} dt = 0.2e^{-0.5}$

$$P(A \mid A \bigcup B) = \frac{P(A)}{P(A)+P(B)} = \frac{1}{1+0.2} = 0.8333$$

Homework Solution 29.8.13

★★★★★ Difficulty



The most difficult task is to figure out what $g(t)$ means. $g(t)$ is the EPV of the single claim at t. A claim occurs when state 0 or 1 moves to state 2.

$g(t) = 10000 e^{-\delta t} ({}_tp_x^{00} \mu_{x+t}^{02} + {}_tp_x^{01} \mu_{x+t}^{12})$

$g(3) = 10000 e^{-3\delta} ({}_3p_x^{00} \mu_{x+3}^{02} + {}_3p_x^{01} \mu_{x+3}^{12})$. You are already given ${}_3p_x^{01} = 0.4174$.

${}_3p_x^{00} = {}_3p_x^{\overline{00}} = \exp\left(-\int_0^3 (\mu_{x+t}^{01} + \mu_{x+t}^{02}) dt\right)$

$= \exp\left(-\int_0^3 (0.20 + 0.10t + 0.05 + 0.05t) dt\right) = e^{-1.425}$

${}_3p_x^{00} \mu_{x+3}^{02} + {}_3p_x^{01} \mu_{x+3}^{12} = e^{-1.425}(0.05 + 0.05 \times 3) + 0.4174(0.15 + 0.01 \times 3^2) = 0.148276$

$g(3) = 10000 e^{-0.02 \times 3} 0.148276 = 1396.4108$

You are given:

(i) Until a scientist is eaten, they suspect nothing, so $\mu_t^{01} = 0.01 + 0.02 \times 2^t$, $t > 0$

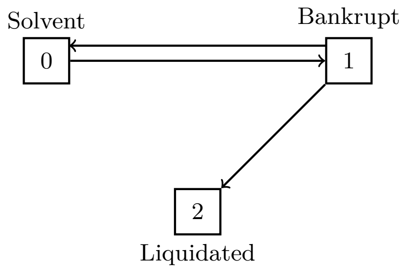
(ii) Until a scientist is eaten, they suspect nothing, so the tyrannosaur may come across two together and eat both, with $\mu_t^{02} = 0.5\mu_t^{01}$, $t > 0$

(iii) After the first death, scientists become much more careful, so $\mu_t^{12} = 0.01$, $t > 0$

Calculate the probability that no scientists are eaten in the first year.

Homework 29.8.15

(Exam MLC: Fall 2012 Q16) You are evaluating the financial strength of companies based on the following multiple state model:



For each company, you assume the following constant transition intensities:

(i) $\mu^{01} = 0.02$

(ii) $\mu^{10} = 0.06$

(iii) $\mu^{12} = 0.10$

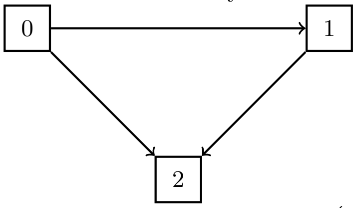
Using Kolmogorov's forward equation with step $h = 1/2$, calculate the probability that a company currently Bankrupt will be Solvent at the end of one year.

Homework 29.8.16

The mortality is $S(x) = 1 - \frac{x}{100}$, $0 \le x \le 100$. Use Kolmogorov's forward equation and the step size $h = \frac{1}{12}$. Calculate the probability that age 40 survives at least two months.

Homework Solution 29.8.14

★★★★☆ Difficulty



$${}_1p_x^{00} = {}_1p_x^{\overline{00}} = \exp\left(-\int_0^1(\mu_{x+t}^{01}+\mu_{x+t}^{02})dt\right) = \exp\left(-\int_0^1 1.5\mu_{x+t}^{01}dt\right)$$

$$\int_0^1 1.5\mu_{x+t}^{01}dt = \int_0^1 1.5(0.01 + 0.02\times 2^t)dt$$

$$= 1.5\left(0.01t + \frac{0.02\times 2^t}{\ln 2}\right)\Big|_0^1 = 1.5\left(0.01 + \frac{0.02}{\ln 2}\right) = 0.05828$$

$${}_1p_x^{00} = e^{-0.05828} = 0.9434$$

Homework Solution 29.8.15

★★★★☆ Difficulty

$$\frac{d}{dt}{}_tp_x^{10} = {}_tp_x^{11}\mu_{x+t}^{10} - {}_tp_x^{10}\mu_{x+t}^{01} = 0.06{}_tp_x^{11} - 0.02{}_tp_x^{10},$$

$$\frac{d}{dt}{}_tp_x^{10}\Big|_{t=0} = 0.06(1) - 0.02(0) = 0.06,$$

$$\frac{d}{dt}{}_tp_x^{11} = {}_tp_x^{10}\mu_{x+t}^{01} - {}_tp_x^{11}(\mu_{x+t}^{10} + \mu_{x+t}^{12}) = 0.02{}_tp_x^{10} - 0.16{}_tp_x^{11},$$

$$\frac{d}{dt}{}_tp_x^{11}\Big|_{t=0} = 0.2(0) - 0.16(1) = -0.16,$$

$${}_hp_x^{10} \approx {}_0p_x^{10} + h\frac{d}{dt}{}_tp_x^{10}\Big|_{t=0} = 0 + 0.06/2 = 0.03$$

$${}_hp_x^{11} \approx {}_0p_x^{11} + h\frac{d}{dt}{}_tp_x^{11}\Big|_{t=0} = 1 - 0.16/2 = 0.92$$

$$\frac{d}{dt}{}_tp_x^{10}\Big|_{t=h} = 0.06(0.92) - 0.02(0.03) = 0.0546,$$

$${}_{2h}p_x^{10} \approx {}_hp_x^{10} + h\frac{d}{dt}{}_tp_x^{10}\Big|_{t=h} = 0.03 + 0.0546/2 = 0.0573$$

Homework Solution 29.8.16

★★★★☆ Difficulty

For a simple alive-dead model, the KM forward equation is ${}_{t+h}p_x \approx {}_tp_x(1 - h\mu_{x+t})$. Set $h = \frac{1}{12}$ and $t = 0$. Notice that $\mu_x = \frac{1}{100-x}$ and ${}_0p_x = 1$.

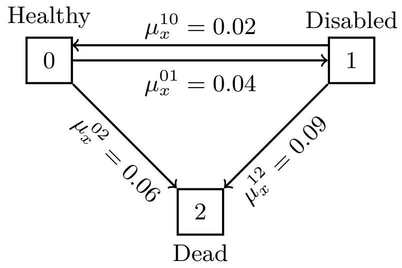
$${}_{\frac{1}{12}}p_{40} \approx {}_0p_{40}(1 - h\mu_{40}) = 1\left(1 - \frac{1}{12}\times\frac{1}{100-40}\right) = 0.998611$$

$${}_{\frac{2}{12}}p_{40} \approx {}_{\frac{1}{12}}p_{40}(1 - h\mu_{40+\frac{1}{12}}) = 0.998611\left(1 - \frac{1}{12}\times\frac{1}{100-40-\frac{1}{12}}\right) = 0.997222$$

The true value is $\frac{S(40+\frac{2}{12})}{S(40)} = \frac{1 - \frac{40+\frac{2}{12}}{100}}{1 - \frac{40}{100}} = \frac{60 - \frac{2}{12}}{60} = 0.997222$

29.9 today's challenge

An insurance company uses the following continuous Markov model for pricing a combined 20-year disability, annuity, and life insurance contract.



The policy is issued to a healthy life age (x).

The transition intensities are constants for all ages.

$\delta = 0.05$

Calculate the EPV of each benefit separately.

Note. Not all benefits can be valued exactly. If there's no exact numerical solution to an EPV, just write down the integral form for that EPV.

(A) 1000 per year payable continuously while the insured is healthy

(B) 1000 per year payable continuously while the insured is healthy but no payment is made if the insured is healthy after recovering from disability

(C) 1000 payable immediately when the insured is disabled

(D) 1000 payable immediately when the insured is disabled for the 1st time

(E) 1000 payable immediately upon death or disability but no death benefit is paid if there's a prior disability claim

(F) 1000 per year payable continuously while the insured is disabled

(G) 1000 per year payable continuously throughout the 1st period of disability

(H) 1000 per year payable continuously throughout the 1st period of disability subject to a 6-month waiting period

(I) 1000 payable at the moment of death and an additional 500 if the insured was disabled at death

(J) a 1000 no-claim bonus payable at the end of the term if there's no death or disability claim during the term